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# CHEBYSHEV'S Polynomial Approximation of an N-Warehouse Stock Allocation Model in Dynamic Programming

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***Abstract* – Optimization of stock allocation is an important aspect of the manufacturing and distribution outfits to ensure minimization of cost and maximization of profit. Polynomial approximation has been applied to obtain better allocations while the chebyshev polynomial approximation is applied. The Remez algorithm, a successive approximation technique that provides best chebyshev approximation is used on resulting continuous functions. Relevant theorems and illustrative example to express the system are included. It is shown that the best approximate is unique.**

***Keywords* – Approximation, Chebyshev Polynomial, Dynamic Programming, Model, Optimization, Stock Controle.**

## PREAMBLE/MOTIVATION

Approximation is the act of estimating a number or an amount. The approximate is almost correct and may not be exact [1]. It is a number which is taken as a close estimate of another number or the method of finding such approximate number [4]. Approximation arises due to the difficulty in obtaining the exact area/volume or dimension of some objects. In addition, the inability of man to fortel the future accurately as a result of his fallibility gives birth to estimation of quantities.

In stock allocation, we may not be able to tell the exact volume of demand of any commodity is a given place at a particular time. The exact costs that may be incurred in the movement /supply of the goods may not be known. This gives rise to the need for approximation so that cost would be minimized. This gives rise to the need for approximation in stock allocation of goods.

The question to be addressed is how close to the actual solution or volume of allocation is the approximate? There are many methods of approximation but some provide better approximates and are more suitable for some situations. Other methods of approximation are: Linear Programming, Fourier, rational approximation, Legendre Polynomial etc.

Stock control is a means by which materials of the correct quantity are made available as at and when required with due regard to the economy in storage and ordering costs, purchase price and working capital. [7].

The problem of effective stock allocation is one that should be handled properly to minimize cost. Customer service has become an important dimension of competition along with price and quality. In order to maintain a company's current customers and acquire new ones, prompt services is always considered for which the first requirement is to have goods readily available. [8].

## I. INTRODUCTION

A polynomial is an expression constructed from variables and constants using the operations of addition, multiplication, subtraction and division. It is an expression of the form $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_n x + a_o$ where $a_n \neq o$. A polynomial function is a function that can be defined by evaluating a polynomial. A function $f$ of one argument is called a polynomial function if it satisfies $f(x) = a_n x^n + a_n x^{a-1} + a_{n-2}x^{n-2} + \cdots + ax + a_o$ for all argument $x$ where $n$ is a non-negative integer and $a_i \big/ i = 0,1,2 \ldots, n$ are constant coefficients. [2]. Polynomial functions are those functions whose values can be calculated by putting the value of the independent variable in a polynomial

Determination of polynomial coefficients requires solution of complicated system of equations. It is possible to avoid such problems by using orthogonal Chebyshev polynomials. This is a method of approximation where the maximum difference between value of function and value calculated from polynomials is minimized [3].

Stocking is an act of keeping goods in a store or warehouse/depot so as to make it available on demand to users. The act of stocking goods to satisfy future demand is vital to the manufacturing and distribution organizations [9]. This act has its associated costs both for keeping and not keeping stock. This work aims at developing the appropriate cost function that balances the total cost resulting from overstocking or under stocking. The major objective of stock allocation models is to obtain an inventory level that minimizes the sum of the storage cost, holding cost and other associated costs. [6]. The dynamic programming method is recursive method and the Chebyschev's polynomial of the first kind is defined by a recursive relation [11].

Dynamic programming is an optimization technique for particular classes of backtracking algorithm while programming refers to the technique of filling a table with values computed from other values. Dynamic programming is applicable to problems exhibiting the properties of overlapping subproblem and optimal structure.[7] Expansion of function in terms of orthorgonal polynomials are very useful as they have good convergence properties. The Chebyshev polynomial is an orthogonal polynomial with

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recursion and can be applied to solve dynamic programming problems. [4].

## II. The Chebyschev's Polynomial Approximation

Chebychev's polynomial is a sequence of orthogonal polynomials which are related to De-moivre's formula and which can be defined recursively. The Chebyshev's polynomials $T_n$ are polynomials of degree n. The Chenyslev's polynomial of the first kind are defined by the recursive relation.

$T_o(x) = 1$

$T_1(x) = x$

$\vdots \quad \vdots$

$T_{n+1}(x) = 2x\,T_{n-1}(x) - T_{n-1}(x).$ [4]

The general approximation problem states thus:

Let $f$ be an element and $S$ a subset of a normed linear space $X$, approximation theory seeks to find an elements of $S$ which is as close as possible to $f$; i.e. to find an element $S^*$ of $S$ such that

$\| f - S^* \| \leq \| f - S \| \quad \forall\, s\epsilon S$

$S^*$ is called the best approximation of $f$ from $S$ relative to the given norm.

The Chesyshev's approximation in particular states that:

If $P_n$ is the collection of all polynomials whose degree is at most $n$ and $f$ be a continuous function on the interval $[a,b]$. The polynomial $P$ is said to be the best approximation to $f$ from $p_n$ if $P^{\epsilon}P_n$ and

$\max_{x\epsilon\,[a,b]} \left|f(x) - P^{(x)}\right| \leq \max_{x\epsilon\,[a,b]} |f(x) - q(x)| \quad \forall q\epsilon P_n$ [5].

### 2.1 Properties of Chebyshev's Polynomial

i) Recursion formula; $T_o(x) = 1,\ T_1(x) = x = xT_o(x),$
$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \geq I$

ii) The leading coefficient is $2^{n-1}$ for $n \geq 1$ and 1 for $n = o$

iii) Symmetric property; $T_n(-x) = (-1)T_n(x)$

iv) $T_n(x)$ has $n$ – zeros in [-1, 1] given by
$x_k = cos\left(\frac{2k+1}{n}\pi\right),$
$k = o, 1, \dots n-1.$

v) Orthogonality property; Set
$(f,y) = \int_{-1}^{1} f(x)\, g(x)\, (1-x^2)^{-\frac{1}{2}},$ then

$$(T_i, T_j,) = \begin{cases} O \ \ if \ \ i \neq \ \ j \\ \frac{1}{2}\pi \ \ if \ \ i = j \neq O \\ \pi \ \ if \ \ i = j = O \end{cases} \qquad \text{i.e continuous case}$$

$(f,g) = \sum_{k=o}^{n} f\ (x_k)\, g(x_k)$ where $\{x_k\}$ are the zeros of $T_{m+1}^{(x)}$

Then for $0 \leq i \leq m, o \leq j \leq j \leq m, (T_i, T_j)$

$$= \begin{cases} o \ \ if \ i \neq \\ \frac{1}{2}m+1) \ \ if \ i = j \neq O \\ m+1 \ \ if \ \ i = j = O \end{cases}$$

(Discrete case)

vi) Minimax property; Of all $nth$ degree polynomials with leading coefficient $1, 2^{1-n}T_n$ has the smallest maximum norm in [-1, 1]. The value of its maximum norm is $2^{1-n}$.

## III. Algorithm/Theorem for Chebyshev's Polynomial Approximation [9]

The Remez algorithm is a successive approximation technique for computing the coefficients of the polynomial in $P_n$ which provides the best Chebyshev approximation to a given continuous function $f$. The general approach is to find the best approximation to $f$ on a set of $n+2$ points in such a manner that the successive best approximation on the finite sets approaches the best approximation over the whole interval.

Before stating the algorithm the following assertions /theorems will be stated as they preclude it.

i) Let $x_0 < x_1 < \cdots < x_{n+1}\ be\ n+2$ distinct numbers. Then the system of equations.
$f(x_k) - \sum_{j=o}^{n} a_j\, x_k^j = (-1)^k{}_\lambda \quad k = 0, 1, 2, \dots, n+1$
has a unique solution in $a_0, a_1, a_2, \ \dots .. a_{n,} \lambda.$

ii) Suppose $f\epsilon\ c[a,b],\ f^1$ exists on $(c,b)$ and $a_1, < a_2 < \cdots < a_m$ are points of $[a,b]$.
If $(-1)^k f(a_k) \geq 0, 1 \leq k \leq m,$ then $f$ has at least m -1 zeros on $[a_1, a_m]$

iii) Suppose $T = \{x_0, x_1, x_2, \dots x_{n+1}\}$ is a set of n+2 points indexed, so that $\{x_0 < x_1 < \cdots < x_{n+1}$ and $f$ is a function defined on $T$.
Let $a_0, a_1, a_2 \dots, a_n,\ \lambda$ be the unique solution to $f(x_k) - \sum_{j=o}^{n} a_j\, x_k^j = (-1)_n^k\, \lambda,$ and define $p$ by $p(x) = \sum_{j=o}^{n} a_j\, x^j$
Then $P$ is unique best approximation to $f$ on the set $T$.

iv) Let $f$ be continuous on $[a,b]$, and let $T = \{x_0, \dots < x_{n+1}$ be a set of n+2 distinct points in $[a,b]$ indexed so that $x_0 < x_1 < \ \dots\ < x_{n+1}.$
Suppose $p\epsilon\ p_n$ and $f - p$ alternates sign at the points $x_0, x_1, .. < x_{n+1}.$

Then

$$E_n(f) \geq E_{n,T}^{(f)} \geq \min_{0 \leq j \leq n+1} \left|f(x_j) - p(x_j)\right|$$

The algorithm now proceeds as follows

i) Choose a set $T_0 = \{x_0^0,\ x_1^0, \dots x_{n+1}^0\}$ of $n+1$ points in $[a,b]$

ii) Solve the system of equations

$$f\ (x_k^0) - \sum_{j=o}^{n} a_j^o\ \ (x_k^0)^j = (-1)^k \lambda_o, \qquad k = 0, 1, \dots, n+1$$

iii) For $a_0^0,\ a_1^0, \dots .a_n^0$ and $\lambda_o$

(if $\lambda_o, = 0$ choose a new set of points for $T_o$),

Now proceed inductively having chosen a set of points $T_m = \{x_0^m,\ x_1^m, \dots\ x_{n+1}^m\}$,

Let $\lambda_m = f(x_0^m) - P_m, (x_0^m)$,

*Let* $P_m$ *be the best approximation of* $f$ *on* $T_m$ *and Then*

Choose a new set

$T_{m+1} = \{x_0^{m+1}, x_1^{m+1} \dots .\ x_{n+1}^m\}$ so that the following conditions are satisfied.

i) $x_0^{m+1} < x_1^{m+1} < \cdots < x_{n+1}^{m+1}$

ii) $|f(x_k^{m+1} - p_m(x_k^{m+1})| = \| f - P_m \|$ *for some* $K$,

iii) $f - P_m$ *alternates sign at the points of* $T_{n+1}$ *and*

$|f(x_k^{m+1}) - p_m(x_k^{m+1})| \geq |\lambda_m|, \quad k = 0, 1, \dots .. n+1$

*if* $|\lambda_m| = \| f - p_m \|$, *then* $P_m$ is the best approximation.

## IV. Theorems

In problems of approximation, the convergence of function play important role. This is because the approximate value must be very close (almost exactly) to the actual solution. The uniqueness of the solution is also important. This section will consider theorems that assert the quality, unbiasedness and consistency of the approximates if it exists.

*4.1.* Let $f \epsilon [-1,1]$ *and* let $S_n$ be the best approximate to $f$ from $P_n$ relative to $\|f\| = \left[\int_{-1}^{1} \frac{f^2(x)d_x}{\sqrt{1-x^2}}\right]^{\frac{1}{2}}$, then

$$\lim_{n \to = \infty} \|f - S_n\| = o \qquad [6]$$

*Proof:*

Let $\| \quad \|$ and $\| \quad \|_0$ be the norms defined by $\|f\|$ above and

$$\|f\| = \left[\int_{-\pi}^{\pi} \frac{f^2(x)d_x}{\sqrt{1-x^2}}\right]^{\frac{1}{2}} \text{ Respectively.}$$

Let $S_n\ (\theta) = \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos j\theta$

Making, the change of variable $x = \cos\theta$ and using

$f(x) - \frac{a_0 T_o(x)}{2} - \sum_{j=1}^{n} a_j T_j(x) = g(\theta) - \frac{a_o}{2} - \sum_{j=1}^{n} a_j \cos j\theta$

gives

$\| f - S_n \|^2 = \int_{-1}^{1} \frac{|f(x) - S_n(x)|^2 d_x}{\sqrt{1-x^2}} = \int_o^{\pi} [g(\theta) - S_n(\theta)]^2$ do

$= \frac{1}{2} \| g - S_n \|_0^2$

Recall that if $a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos jx d_x$ and

$b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} fx\, S_n jx\ d_x \qquad j - 1, 2, \dots n$

and

$S_n(x) = T^{(x)} = \frac{a_0}{2} + \sum_{j=1}^{n} (a_j \cos jx + b_j \sin jx),$ *then*

$\lim_{n \to = \infty} \|f - S_n\| = o$

Hence $\frac{1}{2} \| g - S_n \|_0^2 = o$

*4.2.* If $f$ is continuous on $[a, b]$, then a best approximation to $f$ from $P_n$ exists [9]

*Proof:*

Consider the bounded set of coefficients

Let $E = \inf_{P \epsilon P_n} \| f - P \|$

Recall that the zero polynomial belongs to $P_n$, hence $E \leq \|f\|$

Let $S = \{P \epsilon P_n : \|f - p\| f\|\}$.

Then

$\inf_{P \epsilon S} \| f - p \| = \inf_{P \epsilon p_n} \| f - p \| = E$ since $E \leq \| f \|$

Now if $P \epsilon s$, then

$\| p \| = \| p - f + f \| \leq \| p - f \| + \| f \| \leq 2 \| f \|$

Now let $x_0 = a, x_1 = a + h, \dots\ x_n = a + nh = b$ where $h = \frac{(b-a)}{n}$

Using the Lagrange interpolation formula, we have

$p(x) = \sum_{j=0}^{n} p(x_j) l_j(x)$

where $l_j(x) = \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{(x - x_k)}{x_j - x_k}$

Let $C = (C_0,\ C_1,\ \dots .\ C_n)$ *and* $(\propto_{j0},\ \propto_{j1}, \dots . \propto_{jn})$ be the coefficient vectors associated with $p$ and $l_j$, respectively.

Now for $i = 0, 1, 2, \dots . n$

$$C_i, = \sum_{j=o}^{n} p(x_j)\ \propto_{ji}$$

$$\text{Let, } M = \max_{o \leq i \leq n} \sum_{j=0}^{n} |\propto_{ji}|$$

Then

$$|\ C_1| \leq \sum_{j=o}^{n} |p(x_j)\ \| \propto_{ji}\ | \leq 2 \| f \| \sum_{j=o}^{n} |x_{ji}| \leq 2m \| f \|$$

If we let

$\| c \| = \sqrt{c_0^2 + c_1^1 + \cdots + c_n^2}$ be the usual Euclidean norm, then $P \epsilon S$ implies that

$\| c \| \leq 2m \| f \| \sqrt{n+1}$ and

Let $T$ be the set of polynomials in $P_n$ whose coefficient vectors are in u.

Since $S \subset T \subset P_n$ and $\inf_{p \epsilon s} \| f - p \| = \inf_{p \epsilon p_n} \| f - p \| = E$ holds we have

$E \leq \inf_{p \epsilon T} \| f - p \| \leq \lim_{P \epsilon s} \| f - p \| = E$

Thus $E = \inf_{P \epsilon T} \| f - p \|$

For each $c \epsilon$ ∪, let $\varphi(c) = \| f - p \|$, where $p$ is the polynomial with coefficient vector $c$,

$\Longrightarrow E = \inf_{c \epsilon \cup} \Phi(c)$.

But $\varphi$ is continuous on the compact set u.

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Hence there is a vector $c^* \epsilon\, u$ such that

$\varphi\,(c^*) = \inf_{C \epsilon \cup} \varphi\,(c) = E$

Then the polynomial $P^*$ *whose* coefficient is $c^*$ is a best approximation of *f*.

*4.3.* If $f \epsilon [a.b]$, then the best chebyshev approximation to $f$ from $P_n$ is unique.[3]

*Proof:* Let $p_1$ *and* $p_2$ be best approximations to $f$ from $P_n$, $P_{1,}\ P_2 \epsilon P_n$

Now define $P_0 = \frac{(p_1+p_2)}{2}$ *and*

Let $E = \| f - P_1 \| = f - p_2 \|$

Then

$E \le \| f - P_0 \| = \| \frac{f-p_1}{2} + \frac{(f-p_2)}{2} \| \le \| \frac{f-p_1}{2} \| + \frac{\|f-p_2\|}{2} = E$

Thus $P_0$ is also a best approximation to $f$.

Let $x_0 < x_1 < \cdots < x_{n+1}$ be points satisfying the equation.

$$f(x_j) - P_0(x_j) = (-1)^j \delta\, E, j = 0, 1\ \ldots., n+1$$

Then for $j = o, 1, \ldots n+1$

$$E = |f(x_j) - P_0(x_j)\ | = \left| \frac{f(x_j) - P_1(x_j)}{2} + \frac{f(x_j) - P_1(x_j)}{2} \right|$$

But $|f(x_j) - p(x_j)\ |$and $|f(x_j) - p(x_j)\ |$ are both less than or equal to E

$$\Rightarrow f(x_j) - P(x_j) = f(x_j) - P_2(x_j) = (-1)^j \sqrt{E}$$

Thus

$$P_1(x_j) = P_2(x_j), j = 0, 1, \ldots.\ , n+1$$

$\Rightarrow$ $P_1 - P_2$ has n+2 zeros hence

$P_1 = P_2$

## V. Steps to Finding Best Approximation with CHEBYSHEV'S Polynomial (Model)

Given a function $f$ to be approximated, choose another function $p$ to approximate $f$ from $p_n$. i.e $p\ \epsilon\ P_n$

Subtract $p$ from $f$ i.e. $f(x) - p(x) = F(x)$

Obtain $F^{/}(x)$ if it exists.

If the zeros of $F^{I}$ form an oscillation set, then $P$ as a polynomial approximates $f$.

If $|\lambda_m| = \| f - P_m \|$, then $p_m$ is the best approximation if

$$\max_{x \epsilon [a.b]} |f(x) - p(x)| < \max_{x \epsilon [a.b]} |f(x) - q(x)|$$

where

$$q(x)\ \epsilon\ P_n(x)$$

## VI. Illustration

The table below is the stock allocation of a company to their six warehouses. Obtain the best approximation of allocation to ensure profitability of the company.

| Warehouse *(x)* | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Allocation (000 units) *f(x)* | 130 | 293 | 8 | 556 | 943 | 43 |

The corresponding function to the allocation table above from geometry or cubic regression is

$$f(x) = 4x^3 + 2x^2 + x + 1.$$

Now, $f(x) = 4x^3 + 2x^2 + x + 1,$

Use $p(x) = 2x^2 + 4x + 1$ as the best approximate for *f(x)*

$$\therefore f(x) = f(x) - p(x)$$

$$f(x) = 4x^3 + 2x^2 + x + 1 - (2x^2 + 4x + 1 +$$

$$= 4x^3 - 3x$$

$$F^{/}(x) = 12x^2 - 3 = 0$$

$$= 3(4x^2 - 1)$$

$$4x^2 - 1 = 0$$

$$\Rightarrow x^2 = \pm \frac{1}{4}$$

$$x = \pm 0.5 \qquad i.e. \quad \lambda = \pm 0.5\ \epsilon\, [-1,1]$$

i.e $F^{/}$ is zero at $\pm$ 0.5

$\| f \| = \max\, [\,|f(-1)|,\ |F(-0.5)|,\ |F(0.5)|, |F(1)|] = 1$

We note that $-1, -0.5, 0.5$ *and* $1$ form an oscillation, set of four points

$\therefore$ $P$ is the best polynomial approximate for $f$ from $P_2$.

To show that $p$ is the best approximation, we try two other approximations in $P_n$ i.e.

$P_a(x) = 4x^2 + 2x + 1$ and $P_b(x) = x^2 + 2x + 1$

Now

$$f(x) - P_a(x) = F_1(x)$$
$$F_a(x) = 4x^3 - 2x^2 - x$$

$$F_a^{/}(x) = 12x^2 - -4x - 1$$

$$F_a^{/}(x) = 0$$

$$\Rightarrow \lambda_1 = -0.1667 \text{ and } 0.5\ \epsilon[-1,1]$$

$$F_b(x) = f - P_b(x) = 4x^3 + x^2 - x$$

$$= F_b^{/}(x) = 12x^2 + 2x - 1$$

$$F_b^{/}(x) \text{ is zero at } -0.342 \text{ and } 0.217$$

$$\lambda_2 = -0.343 \text{ and } 0.217\ \epsilon\, [-1,1]$$

The zeros of $F_a^{/}$ and $F_b^{/}$ do not form oscillation sets though they are all approximations of $f$. Hence $\lambda\ \pm$ 0.5 i.e $p$ is the best approximation. Furthermore $|\lambda| = \pm 0.5$ applied in $|f(x) - p(x)\|$ shows that

$$\max_{x \epsilon [-1,1]} |f(x) - P(x)| < \max_{x \epsilon (-1,1)} |f(x) - P_a(x)|$$
$$< \max_{x \epsilon (-1,1)} |f(x) - P_b(x)|$$

For $P \epsilon P_n$, $P_a \epsilon P_n$ *and* $P_b \epsilon P_n$

$\therefore$ $P$ is the best approximation to $f$ from $P_n$ relative to the uniform (or chebyshev) norm. It is unique.

## VII. Summary and Conclusion

Stock allocation and polynomial approximation have, been discussed. Chebyshev's polynomial and the properties are stated. The Remez algorithm which is a successive approximation technique for computing the coefficients of the polynomial in $P_n$ is also stated. The Remez algorithm provides the best chebyshev approximation to a given continuous function $f$ [6]

Relevant theorems have been stated with their proofs while some are just stated. An illustration is also included to buttress the .workability of the system.

It has been shown that the best approximation is indeed unique.

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