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Functions in Polar Coordinates to Determine the Kinematic Manipulability in Serial Robots

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***Abstract* – Let nDoF denote a mechanism of n degrees of freedom based on point o_1. The problem considered is: assume that in a neighborhood of point p_0 the vectorial description of the movement of the End-Effector(Ef) is given by the function u. Once the system arrives at $\mathbf{u}(\mathbf{p_0})$, in which direction is the magnitude(the norm) of the velocity vector maximal (or minimal)?. For 2DoF the answer is given by the velocity ellipses at every point. The real problem turns out to be practical calculation of the ellipses. Until now only partial answer has been given via the eigenvalues of A, the matrix of the system and a function of the Jacobian of uat p_0, because the lengths of the semiaxes are functions of them. The problem of finding the sought direction in the workspace has not been practically considered. In this paper a complete procedure is given to calculate the ellipses at every point in that workspace. First it is shown how the introduction of polar coordinates for the eigenvalues calculations greatly simplifies them. The main contribution is a practical procedure, based on geometrical considerations, to calculate the angle of the major semiaxis with the coordinate axis and thus the orientation (or equivalently, the eigenvectors). It is shown that there is no need to recalculated eigenvalues and eigenvectors at each point. It is enough to calculate them at a given set of points and then interpolate to know the corresponding values. The use of the condition number of A gives important insights in the situation. A complete map of ellipses is given for 2DoF.**

***Keywords* – Velocity Vector, Velocity Ellipses, Eigenvalues, Eigenvectors, Polynomial Function, Polar Coordinates, Map of Ellipses.**

I. Introduction

Given a serial system of n degrees of freedom, nDoF, based on $\boldsymbol{o_1}$, the problem here considered is the one of the kinematic performance of the system. Under performance of a system it is first understood its capability to operateunder some optimality requirements. Under kinematic performance it is understood the capability of the system to“choose” a direction at some point $\mathbf{p}$of its trajectory such that the (new) velocity vector represents a “velocity gain”, i.e. this vector has a bigger magnitude (norm) than any other velocity vector with which the system arrives at $\mathbf{p}$.

For the two-degrees-of-freedom case the solution is given by the so-called “velocity ellipses”, Sciavicco andSicilian [2] defined the velocity manipulability ellipses, they represent the reaction of the system to arbitrarily change of position and orientation of the Ef. Tsai [3] utilized the velocity manipulability ellipses to measure the velocity required in the articulate joints given a position and the same magnitude of velocity in all direction in the workspace.

The problem turns then out to be a practical problem: How to calculate efficiently the ellipse for any point of the workspace?.

Only partial solutions can be found in the bibliography. The ellipses depend on the eigenvalues and eigenvectors of the gradient of the transformation, and the eigenvector procedure is precisely a method to find the needed ones. The first contribution of this paper is to show how the introduction of some basic concepts of complex numbers simplifies some calculations. The complementary problem of finding the orientation of the axis of the ellipse has not been treated. In this paper a complete practical solution to the complete problem is given and practically demonstrated giving a complete map of ellipses for 2DoF.

Statement of the problem and of the given solutions

Consider 2DoF, assume the “maximal” length of the system be $\mathrm{L} = \mathrm{l}_1 + \mathrm{l}_2$, where $\mathrm{l}_1$and $\mathrm{l}_2$are the lengths of the two links of the system. It is here assumed that$\mathrm{l}_1 = \mathrm{l}_2$. The workspace of the system is defined as the set of all points theoretically attainable by its Ef. In this case the workspace is the circle of radius L.

The movement of the Ef is a function u of the two angles, $\boldsymbol{\theta} = (\theta_1, \theta_2)$, where $\mathbf{u}$is a continuous and derivable function. Denote by $\nabla\mathbf{u}(\mathbf{p_0})$ the gradient of $\mathbf{u}$ at the point$\mathbf{p_0} = \mathbf{u}(\theta_{10}, \theta_{20})$, given by the Jacobian$\mathbf{J}$of u with respect to $\boldsymbol{\theta}$**.** Numerically, points in a relatively small neighborhood of a singularity exhibit a bad behavior, that can be analyzed via the kinematic manipulability measure, $w = \left(\det(JJ^T)\right)^{1/2}$, Yoshikawa [4], or with the Condition Number of Jacobian matrix [6],[7], [8].

For any points in the workspace, without singular positions, the problem is then:

Problem 01. Let $\boldsymbol{\theta_0} = (\theta_{10}, \theta_{20})$ and considered $\mathbf{p_0} = \mathbf{u}(\boldsymbol{\theta_0})$. Assume it is possible to change arbitrarily the direction of the Ef when$\boldsymbol{\theta} = \boldsymbol{\theta_0}$ Let $\dot{\mathbf{u}}(\boldsymbol{\theta_0})$denote the velocity vector at $\boldsymbol{\theta_0}$. Denote by$\dot{\mathbf{u}}(\boldsymbol{\theta_0} + \Delta\boldsymbol{\theta})$the velocity vector of the movement in the new direction after the better direction calculable in $\mathbf{p_0}$. In which direction should the Ef move such that the norm of $\dot{\mathbf{u}}(\boldsymbol{\theta_0} + \Delta\boldsymbol{\theta})$attains a maximum?

Solution. It is possible to center an ellipse at every nonsingular point of the workspace such that the segment between $\mathbf{p}$ and every point of the ellipse indicates a possible direction for the system and its length the magnitude or norm of the velocity vector in each case. Supposed $\Delta\boldsymbol{\theta}_{\max}$ and $\Delta\boldsymbol{\theta}_{\min}$ are small displacements of the joints that gives the direction of major and minor semiaxes. Consequently the **maximum velocity gain** is the difference norm of $\dot{\mathbf{u}}(\boldsymbol{\theta}_0 + \Delta\boldsymbol{\theta}_{\max})$ and norm of $\dot{\mathbf{u}}(\boldsymbol{\theta}_0 + \Delta\boldsymbol{\theta}_{\min})$.

The lengths of the major and minor semiaxes of the ellipse can be determine with the eigenvalues of $\mathbf{A} = (\mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1/2}$. And the direction of the principal semiaxes can becalculated with eigenvectors that can be assumed to begeometrically represented by angles. The angle betweenthe major semiaxis and the $\mathbf{x}_1$-axis is denoted by Ψ.

It is shown how to calculate directly the eigenvalues using some notation from complex numbers. The calculation is compared with a numeric procedure.

Then the two main contributions follow. First an efficient method, based on geometrical considerations, to calculate Ψ is presented for any given point in the workspace. Second, it is shown that it is not necessary to make the calculations of the eigenvalues for every point. Giving a relatively small number of points, all other values can be obtained via interpolation. To demonstrate the feasibility of the here presented methodology, a complete map of velocity ellipses is given for 2DoF.

II. Kinematic Manipulability Measure

The typical kinematic problem of velocity is: consider a serial mechanical architecture with nDoF its joints variables are denoted by $\boldsymbol{\theta}_0 = (\theta_1, \theta_2, \ldots, \theta_n)^T$. It is assumed that the position and/or orientation of the Ef can be described with variables, $\mathbf{u} = (u_1, u_2, \ldots, u_m)^{\mathrm{T}} (m \leq n)$. All given with respect to a reference orthogonal coordinate frame. The kinematic relation between $\mathbf{u}$and $\boldsymbol{\theta}$ be given by:

$$\mathbf{u} = \mathbf{f}(\boldsymbol{\theta}) \qquad (1)$$

where $\mathbf{u} \in \mathbb{R}^{\mathbf{m}}$, $\boldsymbol{\theta} \in \mathbb{R}^{\mathbf{n}}$and $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^m$. And the Ef's velocity,$\dot{\mathbf{u}}$is obtained by$\dot{\mathbf{u}} = \frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{u}) = \frac{\mathbf{d}}{\mathbf{dt}}(\mathbf{f}(\boldsymbol{\theta}))$, i.e.

$$\dot{\mathbf{u}} = \mathbf{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad (2)$$

where$\dot{\mathbf{u}} \in \mathbb{R}^{\mathbf{m}}$,$\dot{\boldsymbol{\theta}} \in \mathbb{R}^{\mathbf{m}}$ and $\mathbf{J}(\boldsymbol{\theta}) \in \mathbb{R}^{m \times n}$. The matrix $\mathbf{J}(\boldsymbol{\theta})$isthe Jacobian of $\mathbf{u}$and this will be written as$\mathbf{J}$hereafter.

From equation 2, the inverse kinematic problem is given by,

$$\dot{\boldsymbol{\theta}} = \mathbf{J}^{-1}\dot{\mathbf{u}} \qquad (3)$$

By definition, any $\boldsymbol{\theta}^*$ such that $\det(\boldsymbol{J}(\boldsymbol{\theta}^*)) = 0$ is called asingularity of the kinematic configuration. The singularities represent configurations at which the mobility of the structureis reduced, that is to say, it is not possible to impose an arbitrary motion to the Ef. And in the neighborhood of a singularity, small velocities in the operational space may cause large velocities in the joint space or vice versa [2].

Now if we want to measure the reaction of a manipulator to arbitrarily change Ef's position is necessary to evaluate the transformation characteristics in velocity [2],[3],[5],[7]. This is important because is possible to determinate a be stand worst direction of the Ef in specific position in the workspace. With the best direction at the Ef is possible to obtain most velocity gain and viceversa with the worst direction. Here in this paper we are interested to determine these directions in the workspace and analyzed it.

To evaluate the transformation characteristics can be obtained comparing the velocity at the Ef if the joint velocity is confined to an unit, i.e.

$$\|\dot{\boldsymbol{\theta}}\| = 1 \qquad (4)$$

substitutingequation 3 in 4

$$\dot{\boldsymbol{\theta}}^{\mathrm{T}}\dot{\boldsymbol{\theta}} = 1 \Rightarrow (\mathbf{J}^{-1}\dot{\mathbf{u}})^{\mathrm{T}}(\mathbf{J}^{-1}\dot{\mathbf{u}}) = 1$$

$$\dot{\mathbf{u}}\,\underbrace{(\mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1}}_{\mathbf{i}}\,\dot{\mathbf{u}} = 1 \qquad (5)$$

The equation 5 represent an ellipsoid in $\mathbb{R}^m$.The term "i" of this equation is a positive definite symmetric matrix, $\mathbf{A}$.

Based on Tsai [3] and Sciavicco and Sicilian [2], there are two procedures to determine this transformation characteristics and analyzed it in the workspace of manipulator: a) numerical procedure (NP) and b) eigenvectors procedure (EP).

Numerical procedure is a direct form of obtaining the transformation characteristics of velocities in manipulators (in this case a manipulator of 2DoF). It is given by

a1). The condition given by equation 4 implies that the unitary circle of angular velocities can be given parametrically

$$\dot{\boldsymbol{\theta}}(\phi) = \begin{pmatrix} \dot{\theta}_1(\phi) \\ \dot{\theta}_2(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \end{pmatrix}, \phi \in [0, 2\pi]$$

where$\phi$is divided in N subintervals such that

$$[0, 2\pi] = \left[0, \frac{a}{N}\right] \cup \left(\frac{a}{N}, \frac{2a}{N}\right] \cup \cdots \cup \left(\frac{(N-1)a}{N}, a\right]; a = 2\pi$$

For $\phi_j = \left(\frac{(j-1)a}{N}\right)$ is calculated

$$\dot{\boldsymbol{\theta}}_j = \dot{\boldsymbol{\theta}}(\phi_j);\ j = 1, 2, \ldots N, N+1 \qquad (6)$$

a2). The Ef's velocity is calculated using equations 2 and 6

$$\dot{\mathbf{u}}(\dot{\boldsymbol{\theta}}_j) = \boldsymbol{J}\dot{\boldsymbol{\theta}}_j;\ j = 1, 2, \ldots N, N+1 \qquad (7)$$

a3). Find the maximum and minimum norm of the end-effector's velocities

$$\dot{u}_{\max} = \max_{\dot{\boldsymbol{\theta}}_j}\{\|\dot{\mathbf{u}}(\dot{\boldsymbol{\theta}}_j)\|\};\ j = 1, 2, \ldots N, N+1$$

$$\dot{u}_{\min} = \min_{\dot{\boldsymbol{\theta}}_j}\{\|\dot{\mathbf{u}}(\dot{\boldsymbol{\theta}}_j)\|\};\ j = 1, 2, \ldots N, N+1$$

where $\dot{u}_{\max}$: is interpreted as the length of the major semiaxis,$\dot{u}_{\min}$: is the length of the minor semiaxis of an ellipse and $\dot{\boldsymbol{\theta}}_j$: is the vector design of the manipulator joints.

a4).The vector with major ($\dot{\mathbf{u}}_{\max}$) and minor ($\dot{\mathbf{u}}_{\min}$)norm correspond to the direction of principal axis. These are the the major and minor semiaxis vector respectively.

a5). Let $\dot{u}_{\max}$ and$\dot{u}_{\min}$,

$$\kappa_n = \frac{\dot{u}_{\min}}{\dot{u}_{\max}}$$

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where κ_n is interpreted as the reciprocal of the condition number with the 2 norm.

a6). The orientation angle between the major semiaxis vector and $\mathbf{x}_1 = (1,0)^T$ vector is given by

$$\Psi_n = \arc\cos\left(\frac{(\dot{\mathbf{u}}_{max}^T\,\mathbf{x}_1)}{\|\dot{\mathbf{u}}_{max}\|}\right) \qquad (8)$$

The Eigenvectors procedure is the most direct way to establish the ellipse of transformation characteristics.

b1). With the equation **A** are determined the eigenvalues i.e. $\lambda_1, \lambda_2; \lambda_1 \geq \lambda_2; \lambda_1, \lambda_2 \in \mathbb{R}^+$

b2). Let λ_1 and λ_2, then the lengths of semiaxis of the sought ellipse are given by

$$s_a = \lambda_2^{-1/2};\; s_b = \lambda_1^{-1/2} \qquad (9)$$

b3). The eigenvectors of A are determined solving the equation 10

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v}_i = \mathbf{0};\; i = 1,2 \qquad (10)$$

where $\hat{\mathbf{v}}_1 = \hat{\mathbf{s}}_b$ and $\hat{\mathbf{v}}_2 = \hat{\mathbf{s}}_a$ are the unitary vectors of $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively, that corresponds to the direction of minor and major semiaxis.

b4). The Condition Number of a matrix **A** [9],[10],

$$\kappa = \|\mathbf{A}^{-1}\|\|\mathbf{A}\|;\; \kappa \in [1,\infty) \qquad (11)$$

where $\|\cdot\|$ represent the matrix norm, can be determined with the 2-norm or Frobenius norm. With the purpose of obtaining a range (0,1] is determined the reciprocal of the Condition Number, $\kappa^{-1} = \frac{1}{\kappa}; \kappa^{-1} \in (0,1]$.

b*5).* Given the eigenvector corresponding to the direction of the major semiaxis, $\hat{\mathbf{s}}_a$, the angle Ψ_e with the $\mathbf{x}_1$ axis is given by

$$\Psi_e = \arc\cos(\hat{\mathbf{s}}_a^T \mathbf{x}_1) \qquad (12)$$

b6). The ellipse of linear velocity is build with equation 9, the unitary eigenvectors and parametric equation of ellipse.

For compare the formation of ellipse with both procedure is chosen one point $\mathbf{p} = (x,y)^T = (0.323558, 0.235079)^T$. The mechanical architecture of 2DoF (figure 3) where the links's length are $l_1 = l_2 = 0.25$ and elbow down arm configuration. The units of length are in meters, the time in seconds and the angle in radians.

In table I are shown the values of the major and minor semiaxis vector of linear velocity obtained with both procedure

Table I. Max. and min. values of linear velocity ellipses

	NP Procedure	EP Procedure
$\dot{\mathbf{u}}_{max} = (\dot{u}_x, \dot{u}_y)^T$	(-0.32188,0.31822)	(-0.32197,0.31814)
$\dot{\boldsymbol{\theta}}_{max}^{opt} = (\dot{\theta}_1, \dot{\theta}_2)^T$	(0.87231,0.48896)	(0.87187,0.48975)
$\dot{\mathbf{u}}_{min} = (\dot{\theta}_1, \dot{\theta}_2)^T$	(-0.09314,-0.09434)	(-0.09318,-0.09430)
$\dot{\boldsymbol{\theta}}_{min}^{opt} = (\dot{\theta}_1, \dot{\theta}_2)^T$	(-0.48986,0.87180)	(-0.48975,0.87187)

In figure 1 are shown the unitary circle of joints's velocities and the linear velocities ellipse at the manipulator's Ef, where $\dot{\boldsymbol{\theta}}_{max}^{opt}$ and $\dot{\boldsymbol{\theta}}_{min}^{opt}$ are the vector design in manipulator joint that produced the maximum ($\dot{\mathbf{u}}_{max}$) and minimum ($\dot{\mathbf{u}}_{min}$) norm of Ef's velocities respectively, determined with the numerical procedure for N equal to 3050.

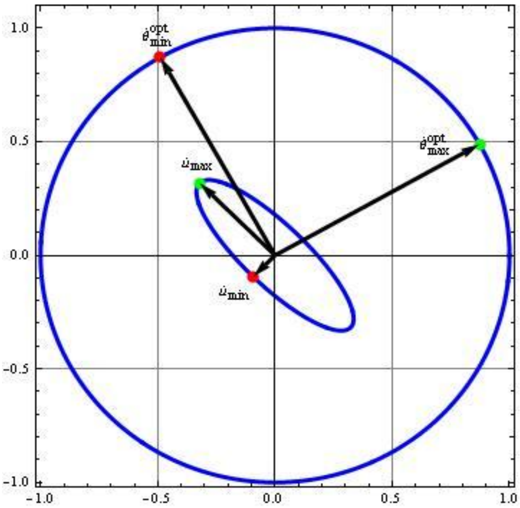

Fig.1. Velocities of manipulator's Ef and in joint articulates.

In figure 2 are shown two ellipses with $\dot{u}_{max}$ and $\dot{u}_{min}$ calculated with both procedures. The EP procedure has the advantage that the more important axes (major and minor semiaxis) are obtained in a direct way and then more precise than NP procedure. For reach a percentage of the relative error (RE) at the Ef's velocity of $RE \leq 0.00176$ was necessary 3050 subintervals. This is a disadvantage of NP procedure, too many calculations are needed to obtain a good precision.

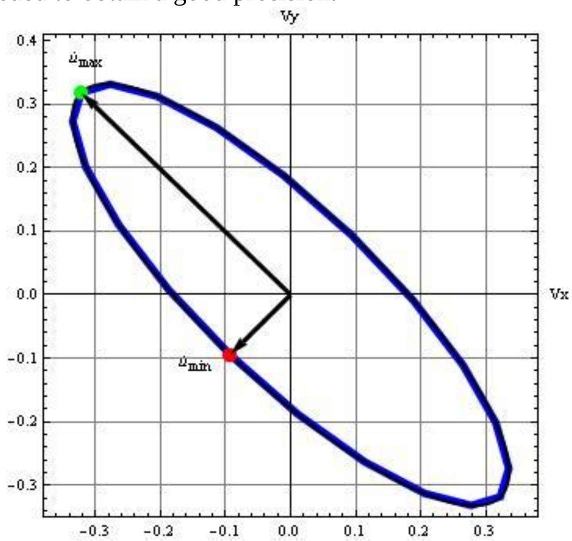

Fig.2. Ef's velocities determined with NP and EP procedure.

III. Determination and Analysis of the Ellipses of Linear Velocity

The objective is the construction of the velocity ellipses at the manipulator's Ef in the workspace and the analysis of the main elements of these with the purpose of obtaining functions in polar coordinates that determined the behavior of this ellipses so much elbow down and

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elbow up arm configuration. The EP procedure will be used.

Denote by $\mathbf{P_{ef}}$ the point occupied by the Ef as a function of $\mathbf{P}$(r, Φ) in polar coordinates and $|\mathbf{P}| = r$. Call M the set of all $\mathbf{P_{ef}}$ admisible for a concrete manipulator.

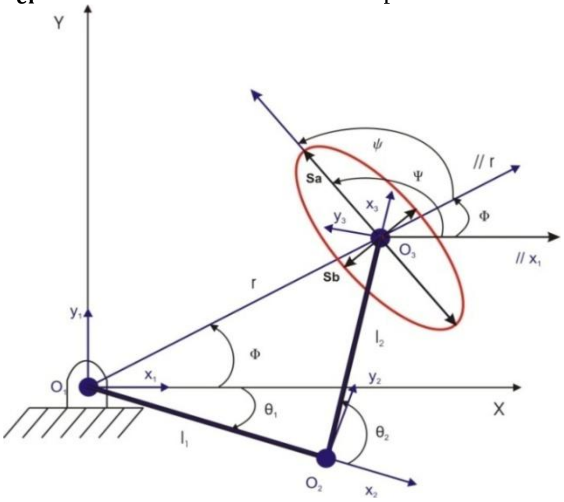

Fig.3. Mechanical architecture and ellipse of linear velocity

Theorem 01. Consider a 2DoF manipulator (fig. 3) fixed on a point o_1. Let $l_1 + l_2 = L$ and N be the set of all points in polar coordinates such that

$$\mathrm{N} = \{(\mathrm{r}, \Phi) |\, 0 < r < L, 0 \le \Phi \le 2\pi\}$$

and assume that the origin of N is coincident with the point o_1. For this case $M \subseteq N$.

Notice that the boundaries of N do not belong to M, if the kinematic relation is given by

$$\mathbf{u} = \mathbf{f}(\boldsymbol{\theta})$$

and thus, as have been seen

$$\dot{\mathbf{u}} = \mathbf{J}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$$

The part 1 expresses a known result (Angeles [1], p. 205).

Part 1. Let λ_{10} and λ_{20} be the eigenvalues of $\mathbf{A} = (\mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1}$ for $|\mathbf{P_0}| = C_1$. Then there is a functional relation between the eigenvalues $(\lambda_{11}, \lambda_{21})$ of $\mathbf{A}$ for any other point, $\mathbf{P_1}$, such that $|\mathbf{P_1}| = C_1$:

$$(\lambda_{11}, \lambda_{21}) = f(\lambda_{10}, \lambda_{20}) \quad (13)$$

Part 1. Let $\mathbf{v_{20}}$ and $\mathbf{v_{21}}$ the eigenvectors of major semiaxis of $\mathbf{A}$ calculated for $|\mathbf{P_0}| = C_1$ and $|\mathbf{P_1}| = C_1$ respectively. And let ψ the angle between the radio vector $\mathbf{r}$ and $\mathbf{v_{20}}$ and Ψ the angle of $\mathbf{v_{21}}$ and $\mathbf{x_1}$-axis. Then there is a functional relation between ψ and Ψ, this is:

$$\Psi = g(\psi) \quad (14)$$

Proof. Part1. The eigenvalues of **A** are canonically computed from the equation,

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \Longrightarrow (\mathbf{A} - \mathbf{I}\lambda)\mathbf{v} = \mathbf{0} \quad (15)$$

one gets λ is the eigenvalues and $\mathbf{v}$ is the eigenvectors of matrix $\mathbf{A}$. Since $\mathbf{v} \neq \mathbf{0}$ then $\det(\mathbf{A} - \mathbf{I}\lambda) = 0$

$$\lambda^2 - \underbrace{\frac{(\csc\theta_2)^2(l_1^2 + 2l_1 l_2 \cos\theta_2 + 2l_2^2)}{(l_1^2 l_2^2)}}_{b}\lambda + \underbrace{\frac{(\csc\theta_2)^2}{l_1^2 l_2^2}}_{c} = 0 \quad (16)$$

This equation 16 is the characteristic polynomial of second grade, the solution is determined if we take the coefficients *a=1*, *b* and *c*,

$$\lambda_1(\theta_2) = \frac{-b + \left(b^2 - 4ac\right)^{1/2}}{2a} \quad (17)$$

$$\lambda_2(\theta_2) = \frac{-b - \left(b^2 - 4ac\right)^{1/2}}{2a} \quad (18)$$

where $\lambda_1(\theta_2) \ge \lambda_2(\theta_2)$. And the singular values are $\sigma_1 = (\lambda_1(\theta_2))^{1/2}$ and $\sigma_2 = (\lambda_2(\theta_2))^{1/2}$.

Taking the reciprocal of Condition Number of **A** with the 2-norm, then

$$\kappa = \frac{\sigma_2}{\sigma_2}, \kappa \in (0,1] \quad (19)$$

From the Inverse kinematic of position, for every point with $|\mathbf{P_1}| = C_1$ the independent variable (θ_2) doesn't change so the ratio, κ, of σ_2 and σ_1 (eq. 19) also doesn't change. Therefore $\lambda_{11}(\theta_2) = \lambda_{10}(\theta_2)$ and $\lambda_{21}(\theta_2) = \lambda_{20}(\theta_2)$.

For the case of elbow down and elbow up arm configuration with the equations 17 and 18, θ_2 is the independent variable. For both configuration **the values of** θ_2 **are complementary**. Therefore the eigenvalues for elbow down arm configuration $\lambda_{11}^{ed}(\theta_2)$ and $\lambda_{21}^{ed}(\theta_2)$, are the same that elbow up arm configuration, $\lambda_{11}^{eu}(\theta_2)$ and $\lambda_{21}^{eu}(\theta_2)$ respectively.

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Proof. Part2. The principal semiaxis correspond the eigenvectors of the matrix A,

$$(\mathbf{A} - \lambda_i \mathbf{I})\mathbf{v_i} = 0;\ i = 1,2 \quad (20)$$

where

$$\mathbf{A} = (\mathbf{J}\mathbf{J}^{\mathrm{T}})^{-1} = \begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} \end{bmatrix} \quad (21)$$

The matrix coefficients are

$$a_{11} = \frac{(\csc\theta_2)^2((l_1\cos\theta_1)^2 + 2l_1 l_2 \cos\theta_1 \cos(\theta_1 + \theta_2))}{l_1^2 l_2^2} + \frac{2l_2^2(\csc\theta_2)^2(\cos(\theta_1 + \theta_2))^2}{l_1^2 l_2^2}$$

$$a_{12} = a_{12} = \frac{(\csc\theta_2)^2(l_1^2 \sin(2\theta_1) + 2l_1 l_2 \sin(2\theta_1 + \theta_2))}{2l_1^2 l_2^2} + \frac{2l_2^2(\csc\theta_2)^2 \sin(2(\theta_1 + \theta_2))}{2l_1^2 l_2^2}$$

$$a_{22} = \frac{(\csc\theta_2)^2((l_1\sin\theta_1)^2 + 2l_1 l_2 \sin(\theta_1)\sin(\theta_1 + \theta_2))}{l_1^2 l_2^2} + \frac{2l_2^2(\csc\theta_2)^2(\sin(\theta_1 + \theta_2))^2}{2l_1^2 l_2^2}$$

Solving the equation 20, are obtained $\mathbf{v_1}$ and $\mathbf{v_2}$. From the matrix 21 the eigenvectors has two independent variables θ_1 and θ_2.

From the figure 3 the orientation of the major semiaxis when $|\mathbf{P_1}| = C_1 = r$ can be calculated through two angles: the first, ψ is the angle between the radio vector $\mathbf{r}$ and major semiaxis $\mathbf{s_a}$, and the second Φ is the angle between the $\mathbf{x_1}$-axis and $\mathbf{r}$. Therefore if we determined an angle ψ_0 calculated for $|\mathbf{P_0}| = C_1$ and $\Phi_0 = 0$. The angle

Ψ_1 between $\mathbf{x_1}$-axis and $\mathbf{v_{21}}$ calculated for $|\mathbf{P_1}| = C_1$ and $\Phi_1 \in (0,2\pi]$, is given by

$$\psi_1 = \psi_0 + \Phi_1, \Phi_1 \in (0,2\pi] \qquad (22)$$

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A. Semiaxeslengths

These elements will be determined in function to polar coordinates (r, Φ),where $r \in (0.0, 0.5)$ and $\Phi \in [0, 2\pi]$ in themanipulators workspace. In the cases of interior and exterior boundaries, the radios are $r_i = 0.0$ and $r_e = 0.5$respectively, in consequence the angles on the second joint in the manipulator are $\theta_2 = \pi$ and $\theta_2 = 0$ and from equations 17 and 18

$$\lim_{\theta_2 \to 0,\pi} \lambda_1(\theta_2) = Indeterminate, \text{and}$$

$$\lim_{\theta_2 \to 0,\pi} \lambda_2(\theta_2) = Indeterminate$$

There fore $\boldsymbol{s_a}$ and $\boldsymbol{s_b}$ are indeterminate in r_i and r_e. And so we use the initial radio $r_0 = 0.0001$ and final radio $r_f = 0.4999$, with a relative error 0.0002.

In figure 4 the positions of the manipulator in elbow down configuration(ed) are shown as the Ef takes positions along a straight line parallel to $\mathbf{x}_1$-axis. From 251 actuallycalculated, 11 werechosenfortheinterpolation.

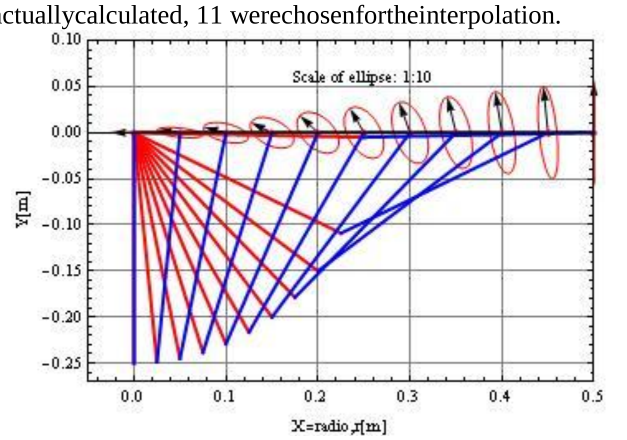

Fig.4. Ellipses of linear velocity on horizontal straight line

In table II the lengths of major (s_a^{ed}) and minor (s_b^{ed}) semiaxes are shown.

Table II: Elements of the Ellipses of linear velocity

r_1[m]	s_a^{ed}	s_b^{ed}	$\psi^{ed}[rad]$	$\psi^{eu}[rad]$
0.00010	0.250000	0.000100	3.14139	0.00020
0.05008	0.250052	0.049818	3.03710	0.10449
0.10006	0.250945	0.097667	2.90334	0.23825
0.15004	0.255833	0.139862	2.69447	0.44712
0.20002	0.272401	0.168242	2.38331	0.75828
0.25000	0.306186	0.176777	2.09440	1.04720
0.29998	0.351127	0.170873	1.91652	1.22507
0.34996	0.400836	0.155892	1.80880	1.33280
0.39994	0.452633	0.132573	1.73386	1.40773
0.44992	0.505476	0.097069	1.67116	1.47043
0.49990	0.558910	0.004472	1.57480	1.56680

In figure 5 the black colored line shows length s_a^{ed} versus radio (on horizontal line). Where the initial radio is r_0 andfinal radio r_f. If we consider the radio r without the singular case, we can obtain a function to describe the evolution of lengths of major semi axis with independent variable, r. Therefore, for this case a polynomial regression is carried out. The determined function is

$$\mathrm{S_a^{ed}(r) = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + a_5 r^5 + a_6 r^6, \forall r \in [r_0, r_f]} \qquad (23)$$

where $\mathrm{a_0} = 0.249816, \mathrm{a_1} = 0.105246, \mathrm{a_2} = -2.34181, \mathrm{a_3} = 12.4696, \mathrm{a_4} = 13.864,$
$\mathrm{a_5} = -98.4433, \mathrm{a_6} = 95.5748.$

The graphics of this function (eq. 23) appears also in figure 5 in red color. Both graphics are almost identic. Their correlation coefficient is R=1.

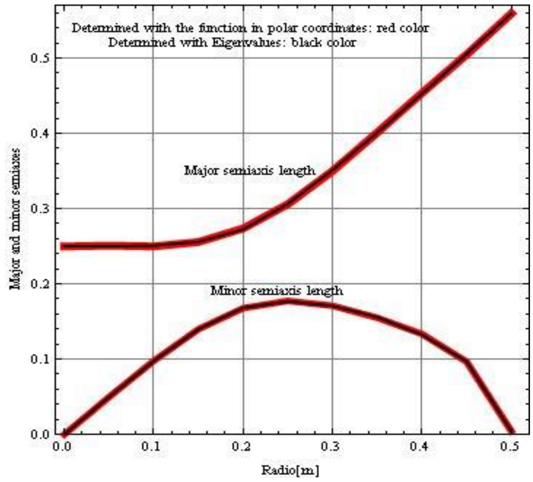

Fig.5. Radio (r) vs major and minor semiaxis length

In the case of the minor semiaxislength (s_b^{ed}), the following equation was obtained after applied a polynomial regression

$$\mathrm{S_b^{ed}(r) = b_0 + b_1 r + b_2 r^2 + b_3 r^3 + b_4 r^4 + b_5 r^5 + b_6 r^6, \forall r \in [r_0, r_f]} \qquad (24)$$

Where $\mathrm{b_0} = 5.15876 \times 10^{-6}, \mathrm{b_1} = 1.04142, \mathrm{b_2} = -1.99285, \mathrm{b_3} = 29.8842, \mathrm{b_4} = -210.016,$
$\mathrm{b_5} = 509.127, \mathrm{b_6} = -418.432.$

It is also shown in figure 5 with red colored line. Their correlation coefficient is $R = 0.999$.

From the theorem 01, the equations 23 and 24 are sufficient to determinate the length of bothsemiaxes in the workspace in both arm configuration.

The equation 19 is graph in concentric circles, $\mathbf{P_{ef}} = \boldsymbol{P}(r_i, \Phi)$, in the workspace. The radio (r_i) showed in table II and $\Phi \in [0, 2\pi]$. The results can be observed in figure 6. The $\kappa^{-1}$is conserved on each circle this implies that the magnitudes of the major and minor semi axis are conserved on each circle as was proved in theorem 01.

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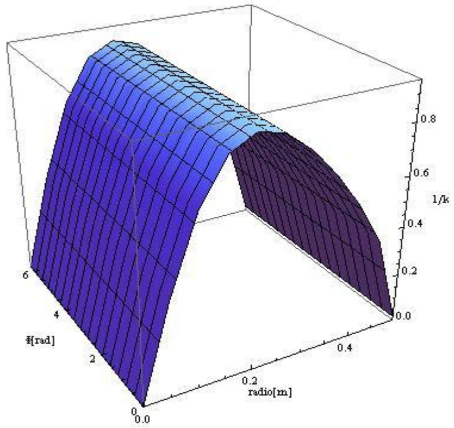

Fig.6. Radio, $r \in [r_0, r_f]$ vs Angle, $\Phi \in [0,2\pi]$ vs κ^{-1}

B. *Orientation of the major semiaxes on horizontal straight line* (ψ)

On the horizontal straight line were determined, with the EP procedure, the orientation angle between the major semiaxis and horizontal $\mathbf{x_1}$ -axis. Thedata that were calculated for elbow down armconfiguration areshown in fourth column of table II.

In the figure 7 is the depicted in red color line the evolution of orientation angle ψ^{ed} versus horizontal radius, $r \in [r_0, r_f]$. We can see that is possible to obtain a function to describe that evolution of orientation angle with independent variable, r. Therefore, for this case a polynomial regression is carried out. The determined function is

$$\psi^{ed}(r) = c_0 + c_1 r + c_2 r^2 + c_3 r^3 + c_4 r^4 + c_5 r^5 + c_6 r^6, \forall r \in [r_0, r_f] \quad (25)$$

where $c_0 = 3.14236, c_1 = -4.00683, c_2 = 67.27816, c_3 = -752.80418, c_4 = 2999.04141, c_5 = -5075.38988, c_6 = 3128.53$.

In the same figure 7 is the depicted in black color line the graphics of the equation 25. Which it is like to the original curve since the correlation coefficient is $R = 0.9998$.

For the case of elbow up arm configuration the orientation angles of major semi axis ψ_i^{eu} are shown in the five column of table II. From these data, we can determined that

$$\psi_i^{eu} = \pi - \psi_i^{ed}; i = 0,1,\dots,n \quad (26)$$

where n=10, radios number.

Therefore the function determined for this configuration is

$$\psi^{eu}(r) = \pi - \psi^{ed}(r), \forall r \in [r_0, r_f] \quad (27)$$

In the figure 7 is also depicted in black color line the graphics of this equation 27.

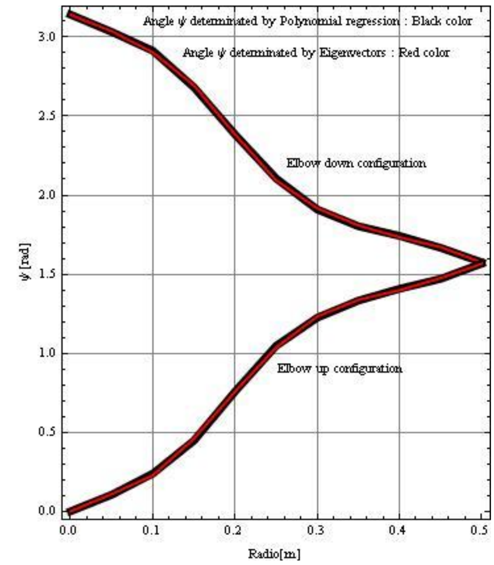

Fig.7. Radio (r) vsOrientation angle (ψ)

C. *Orientation of the major semiaxis in the workspace*

With the same procedure were determined the orientation angles in the workspace in elbow down configuration. This angle Ψ_{ij}^{ed} $i = 0,1,\dots,n$ and $j = 0,1,\dots,cd$(circle discretization) showed in table III were determined in concentric circles with the radios r_i(table II) and the angle $\Phi \in [0, 2\pi]$.

Table III: Orientation of the major semiaxis

$\Phi[rad]$	$\Psi_{0j}^{ed}[rad]$	…	$\Psi_{10j}^{ed}[rad]$
0	3.14139	…	1.57480
π/10	3.45555	…	1.88896
π/5	3.76971	…	2.20312
3π/10	4.08387	…	2.51727
2π/5	4.39803	…	2.83143
π/2	4.71219	…	3.14559
3π/5	5.02635	…	3.45975
7π/10	5.34050	…	3.77391
4π/5	5.65466	…	4.08807
9π/10	5.96882	…	4.40223
π	6.28298	…	4.71639
11π/10	6.59714	…	5.03055
6π/5	6.91130	…	5.34471
13π/10	7.22546	…	5.65887
7π/5	7.53962	…	5.97303
15π/10	7.85378	…	6.28719
8π/5	8.16794	…	6.60135
17π/10	8.48210	…	6.91550
9π/5	8.79626	…	7.22966
19π/10	9.11042	…	7.54382
2π	9.42458	…	7.85798

In the figure 8 is the depicted from blue color lines to black color the evolution of orientation angle Ψ_{ij}^{ed} versus the angle Φ.We can see that the data describe a straight lines.Therefore, for this case we can apply linear regression.The function determined is

$$\Psi^{ed}(\Phi) = d_0 + d_1\Phi, \Phi \in [0,2\pi] \quad (28)$$

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where: d_0 is the initial value ψ_i^{ed} showed in table II. For the general case will be represented by the function $\psi^{ed}(r)$. And $d_1 = 1$. With a correlation coefficient, R=1. Therefore, the equation 28 can be rewriting as,

$$\Psi^{ed}(r,\Phi) = \psi^{ed}(r) + \Phi, \Phi \in [0,2\pi], r \in [r_0, r_f] \quad (29)$$

In the same figure 8 is the depicted from black color line to red color line the graphics of the function $\Psi^{ed}(r,\Phi)$ obtained when the linear regression was applied. Which it is like to the original curve since the correlation coefficient is equal to one.

For the case of elbow up arm configuration. Substituting equation 27 into 28,

$$\Psi^{eu}(r,\Phi) = \psi^{eu}(r) + \Phi, \Phi \in [0,2\pi], r \in [r_0, r_f] \quad (30)$$

In the figure 9 are shown in lines of red color the ellipses map of linear velocity how was obtained with the equations 23,24,25and 29. Since the correlation coefficients approximate to one, these ellipses are very similar.

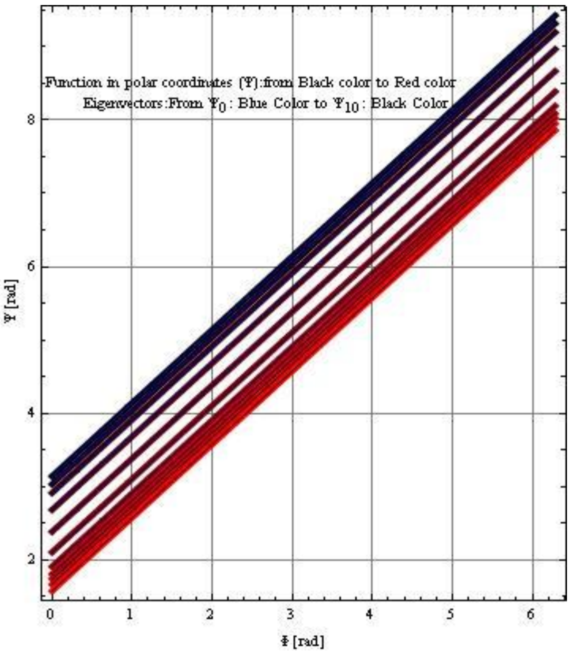

Fig.8. The angle (Φ) vs orientation angle (Ψ)

According with Yoshikawa [5] and, Sciaviccoand Siciliano [2] these ellipses could be a good means for the Analysis, Design and Control of Robot Manipulators

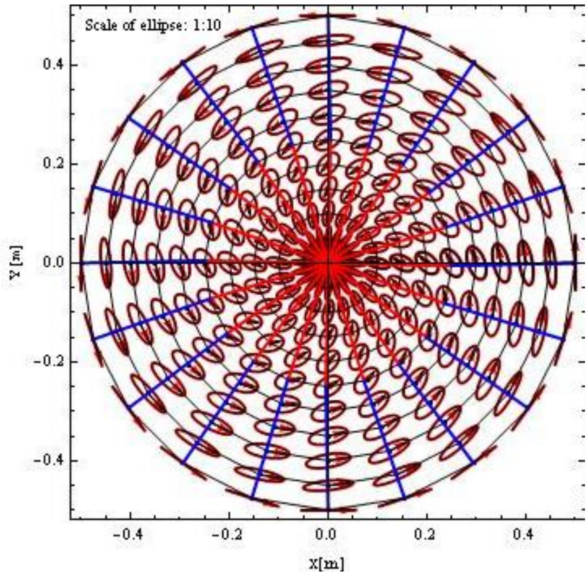

Fig.9. Ellipses of linear velocity in the workspace

IV. CONCLUSIONS

The practical problem of finding the velocity ellipses for any one of the non-singular points of the workspace using an efficient procedure has been completely solved. The use of the kinematic manipulability measure allows a realistic numerical analysis even of points in a neighborhood of singularities.

The problem is solved in two steps: First, for a given set of points $\mathbf{P_{ef}}$, was determined the velocity ellipses and its properties for the manipulator of 2DoF in both arm configuration using the Eigenvector procedure.The numerical method is robust, but requires too many iterations. Second, using polynomials regressions, the functions of major $(S_a(r))$ and minor $(S_b(r))$ semiaxes, and the orientation of major semiaxis in the workspace $(\Psi^{ed}(r,\Phi)$ and $\Psi^{eu}(r,\Phi))$ were obtained.

Therefore, once the Eigenvector method is expressed in the formalism of complex numbers, it is simple enough. Besides, even this simpler procedure does not need to be recalculated for each point. An interpolation using the results for the $|\boldsymbol{P_0}| = C_1$ gives the desired values for any other nonsingular point $|\boldsymbol{P_1}| = C_1$.

The specific contributions of this paper are then:

1. A modification of the usual procedure to obtain the lengths of the principal axis that makes the calculations more efficient.

2. A new procedure to obtain the orientation of the principal axis, equivalent to the calculation of the eigenvectors but geometrically and numerically simpler.

3. In this way it is given: a complete and efficient procedure to calculate the velocity manipulability ellipses for any given point of the workspace together with an evaluation of the numerical performance using the kinematic manipulability measure. This procedure is easily programmable and runs efficiently in small platforms (laptops and similar)

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In conclusion, the practical problem of finding an optimal velocity at a given point, that is, a direction with a máximum gain in the norm of the velocity vector has been totally solved for 2DoF. The procedure and the methodology should also turn out to be a powerful tool for problems with more than two degrees of freedom and its applications.

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