

Spectral Method for the Parabolic Inverse Problem Subject to Temperature Overspecification

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Abstract: In this paper, a numerical method is presented to give the approximate solution of a parabolic inverse problem with a control parameter. The method is derived by expanding the required approximate solution as the elements of Bernstein basis. Using the operational matrix of derivative, the problem can be reduced to a set of algebraic equations. Numerical examples are included to demonstrate the validity and applicability of the technique and the results are compared with the exact solution and other method. The method is easy to implement and produces very accurate results.

Keywords: Inverse Problem, Control Parameter, Parabolic Partial Differential Equations, Bernstein Polynomials.

1. INTRODUCTION

Inverse problem of determination of unknown function in a parabolic differential equation has been treated by many authors [1,2,3,4]. Parabolic inverse problems play a crucial role in applied mathematics and physics. They arise for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion, and control theory [5,6,7,8,9]. In recent years, a lot of attention has been devoted to the study of parabolic inverse problems. Hence, the last 20 years have seen growing attention paid in the literature to the development, analysis, and implementation of accurate methods for the numerical solution of parabolic inverse problems, i.e., the determination of some unknown function $p(t)$ in the parabolic partial differential equations. In this paper we present a direct computational technique for the inverse problem of finding a source parameter $p(t)$ in the following diffusion equation:

$$w_t = w_{xx} + p(t)w + q(x,t), \quad \dots(1)$$

$$0 \leq x \leq 1, 0 < t \leq \tau,$$

with initial condition

$$w(x,0)=f(x), \quad 0 \leq x \leq 1, \quad \dots(2)$$

and boundary conditions

$$w(0,t) = g_0(t), w(1,t) = g_1(t), 0 < t \leq \tau, \quad \dots(3)$$

subject to either the integral overspecification of the function $k(x)w(x,t)$ over the spatial domain,

$$\int_0^1 k(x)w(x,t)dx = E(t), 0 \leq t \leq T, \quad \dots(4)$$

or the overspecification at a point in the spatial domain,

$$w(x_0,t) = E(t), 0 \leq t \leq T, 0 \leq x_0 \leq 1, \quad \dots(5)$$

where f, g_0, g_1, q, k and E are known functions, while the functions w and p are unknown. In addition, it is assumed that, for some constant $h>0$, the kernel $k(x)$ satisfies

$$\int_0^1 |k(x)|dx \leq h, \quad \dots(6)$$

The existence and uniqueness and continuous dependence of the solutions to some kind of these inverse problems are discussed in [10,11,12,13]. The applications of these inverse problems and some other similar parameter identification problems are discussed by several authors. Some finite difference procedures for solution of problem (1)–(6) are given in [14,15,16]. The outline of this paper is as follows. In Section 2, we describe properties of Bernstein polynomials. In Section 3, the presented technique is used to approximate the solution of problem (1)–(6). As a result a set of algebraic equations is formed. Some numerical illustrations are given in Section 4 to show the efficiency of the proposed method. Section 5 ends this paper with a conclusion.

2. THE PROPERTIES OF BERNSTEIN POLYNOMIALS

The Bernstein polynomials of m^{th} degree are defined on the interval $[a,b]$ as [15]

$$B_{i,m} = \binom{m}{i} \frac{(x-a)^i (b-x)^{m-i}}{(b-a)^m}, \quad \dots(7)$$

$$0 \leq i \leq m,$$

where

$$\frac{m}{i} = \frac{m!}{i!(m-i)!} \quad \dots(8)$$

These Bernstein polynomials form a basis on $[a,b]$. For mathematical convenience, we set $B_{i,m}(x) = 0$, if $i < 0$ or $i > m$. These polynomials are quite easy to write down: the coefficients can be obtained from Pascal's triangle. It can easily be shown that each Bernstein polynomials is positive and the sum of all the Bernstein polynomials is unity for all real $x \in [a,b]$, i.e., $\sum_{i=0}^{m=0} B_{i,m}(x) = 1$. A recursive definition also can be used to generate the Bernstein polynomials over $[a,b]$ so that the i^{th} , m^{th} degree Bernstein polynomials can be written as

$$B_{i,m}(x) = \frac{b-x}{b-a} B_{i,m-1}(x) + \frac{x-a}{b-a} B_{i-1,m-1}(x) \dots (9)$$

It is easy to show that any given polynomial of degree m can be expressed in terms of linear combination of the basic functions. See [1] for complete details. In this paper we consider Bernstein polynomials of m^{th} degree interval $[0,1]$. By using binomial expansion of $(1-x)^{m-i}$, we have

$$B_{i,m} = \sum_{k=0}^{m-i} (-1)^k \binom{m}{i} \binom{m-i}{k} x^{i+k}, i = 0,1,2,\dots,m, \quad \dots(10)$$

Suppose that
 $H = L^2[0,1]$ and $\{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\} \subset H$ be the set of Bernstein polynomials of m^{th} -degree and

$$M = span\{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}, \quad \dots(11)$$

and f be an arbitrary element in H . Since M is a finite dimensional vector space, f has the unique best approximation out of M such as $m_0 \in M$, that is

$$\exists m_0 \in M; \forall m \in M, \|f - m_0\|_2 \leq \|f - m\|_2, \quad \dots(12)$$

where $\|f\|_2 = \sqrt{\langle f, f \rangle}$. Since $m_0 \in M$, there exist unique coefficients c_1, \dots, c_m such that

$$f(x) \approx m_0(x) = \sum_{i=0}^m c_i B_{i,m}(x) = c^T \varphi(x) \quad \dots(13)$$

where $\varphi^T = [B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)]$ and $c^T = [c_0, c_1, \dots, c_m]$, and c^T can be obtained by

$$c^T \langle \varphi, \varphi \rangle = \langle f, \varphi \rangle, \quad \dots(14)$$

where

$$\langle f, \varphi \rangle = \int_0^1 f \varphi(x)^T dx = [\langle f, B_{0,m} \rangle, \langle f, B_{1,m} \rangle, \dots, \langle f, B_{m,m} \rangle], \quad \dots(15)$$

and $\langle \varphi, \varphi \rangle$ is a $(m+1) \times (m+1)$ matrix which is said to be the dual matrix of φ and is denoted by Q and will be introduced in the following. Therefore

$$Q = \langle \varphi, \varphi \rangle = \int_0^1 \varphi(x) \varphi(x)^T dx, \quad \dots(16)$$

then

$$c^T = \int_0^1 \{f(x) \varphi(x)^T dx\} Q^{-1} \quad \dots(17)$$

Lemma 1. Suppose that the function $f: [0,1] \rightarrow \mathbb{R}$ is $m+1$ times continuously differentiable, $f \in C^{m+1}([0,1])$, and $M = span\{B_{0,m}(x), B_{1,m}(x), \dots, B_{m,m}(x)\}$

If $c^T B$ be the best approximation f out of M then

$$\|f - C^T B\|_{L^2[0,1]} \leq \frac{k}{(m+1)! \sqrt{2m+3}}, \quad \dots(18)$$

where $K = \max |f^{(m+1)}(x)|$, $x \in [0,1]$, and proof is given in [j]. The previous Lemma shows that the error vanishes as $m \rightarrow \infty$. The derivative of the vector φ can be expressed by

$$\frac{d\varphi(x)}{dx} = D^{(1)} \varphi(x) \quad \dots(19)$$

where D , $(m+1) \times (m+1)$ operational matrix of derivative given by

$$D = A \Lambda V, \quad \dots(20)$$

such that A is a $(m+1) \times (m+1)$ upper triangular matrix where

$$A_{i+1, j+1} = \begin{cases} 0, & \text{for } i > j \\ (-1)^{j-i} \binom{m}{j} \binom{m-i}{j-i}, & \text{for } i \leq j \end{cases}, \quad \dots(21)$$

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & & & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & m \end{pmatrix}, \quad \dots(22)$$

and V is $(m)(m+1)$ matrix can be expressed by

$$\begin{pmatrix} A_1^{-1} \\ A_2^{-1} \\ \vdots \\ \vdots \\ A_M^{-1} \end{pmatrix}, \quad \dots(23)$$

where $A^{-1, k+1}$ is $(k+1)$ th row of A^{-1} matrix. By using Eq. (19), it is clear that

$$\frac{d^k \varphi(x)}{dx^k} = (D^{(1)})^k \varphi(x), \quad \dots(24)$$

where $n \in \mathbb{N}$ and the superscript, in $D^{(1)}$, denotes matrix powers. Thus

$$D^{(k)} = (D^{(1)})^k, \quad k = 1, 2, \dots \quad \dots(25)$$

3. SOLUTION OF THE PROBLEM

Our approach begins with the utilization of the following transformations,

$$u(x, t) = w(x, t) \exp\left(-\int_0^1 p(s) ds\right), \quad \dots(26)$$

$$r(t) = \exp\left(-\int_0^t p(s) ds\right), \quad \dots(27)$$

So, we have

$$w(x, t) = \frac{u(x, t)}{r(t)}, \quad \dots(28)$$

$$p(t) = -\frac{r'(t)}{r(t)}, \quad \dots(29)$$

where $r'(t) = \frac{dr(t)}{dt}$. With this transformation $p(t)$ disappeared, and its role is presented implicitly by $r(t)$. Obviously if we have $u(x, t)$ and $r(t)$ then by using equations (27) and (28), $w(x, t)$ and $p(t)$ can be found. Employing the above pair of transformations (26) and (27), we can write (1)-(6) as follows,

$$u_t = u_{xx} + r(t)q(x, t), \quad 0 \leq x \leq 1, 0 < t \leq T, \quad \dots(30)$$

with

$$u(x, 0) = f(x), \quad 0 \leq x \leq 1, \quad \dots(31)$$

and

$$u(0, t) = r(t)g_0(t), \quad u(1, t) = r(t)g_1(t), \quad \dots(32)$$

$0 < t \leq T$,
subject to

$$\int_0^1 k(x)u(x,t)dx = r(t)E(t), \quad 0 < t \leq T, \quad \dots(33)$$

or

$$u(x_0, t) = r(t)E(t). \quad \dots (34)$$

Suppose $\varphi(x)$ and $\varphi(t)$ are vectors of Bernstein polynomials on $[0,1]$ defined in (13). Now the unknown function $u(x,t)$ in (30) can be approximated as

$$u(x,t) \cong \sum_{i=1}^{n+1} \sum_{j=1}^{m+1} U_{i,j} c_i(x) c_j(t) = \varphi^T(x) U \varphi(t), \quad \dots(35)$$

where the unknown matrix U is $(n+1) \times (m+1)$ and can be shown as

$$\begin{pmatrix} U_{1,1} & \dots & U_{1,m+1} \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ U_{n+1,1} & \dots & U_{n+1,m+1} \end{pmatrix}, \quad \dots(36)$$

Employing 25, we can write

$$u_t(x,t) = \varphi^T(x) U D_t \varphi(t). \quad \dots (37)$$

Also, we have

$$u_{xx}(x,t) = \varphi^T(x) (D_{xx}^2)^T U \varphi(t), \quad \dots (38)$$

Using (37,38) in (30), we obtain

$$\begin{aligned} \varphi^T(x) U D_t \varphi(t) &= \varphi^T(x) (D_{xx}^2)^T U \varphi(t) \\ &+ r(t) q(x,t), \end{aligned} \quad \dots(39)$$

where D_t is $(n+1) \times (n+1)$ and D_{xx} is $(m+1) \times (m+1)$ differential matrix that defined in (20). we now collocate (39) in $(n-1) \times m$ points (x_i, t_j) , $i=2, \dots, n, j=2, \dots, m+1$ as

$$\begin{aligned} R(x_i + t_j) &= \varphi^T(x_i) U D_t \varphi(t_j) - \varphi^T(x_i) (D_{xx}^2)^T U \varphi(t_j) \\ &+ r(t_j) q(x_i, t_j) = 0, \end{aligned}$$

$$i = 2, \dots, N, \quad j = 2, \dots, M+1, \quad (40)$$

For suitable collocation points we use roots of shifted legendre polynomial $[k]$. Collocating (31-34) in $n+1$ points $x_i, i=1, \dots, n+1$ and m points $t_j, j=1, \dots, m$ we gain

$$u(x_i, 0) = f(x_i) \quad i = 1, 2, \dots, n+1, \quad \dots(41)$$

$$u(0, t_j) = r(t_j) + g_0(t_j) \quad j = 1, 2, \dots, n+1, \quad \dots(42)$$

$$u(1, t_j) = r(t_j) + g_1(t_j) \quad j = 1, 2, \dots, m, \quad \dots(43)$$

$$u(x_0, t_j) = r(t_j) + E(t_j) \quad j = 1, 2, \dots, m+1, \quad \dots(44)$$

or instead of (44), we obtain

$$\int_0^1 k(x)u(x, t_j)dx = r(t_j)E(t_j). \quad \dots (45)$$

40 together with (41-45) give a system of equations, which can be solved for the unknown $U_{i,j}$ and r_j .

4. NUMERICAL RESULTS

In this section, we give some results of numerical experiments with methods based on the preceding sections. In this example we will solve the problem (30-34) with the given data (Table 1, 2 and Fig 1, 2)

$$\begin{aligned} \varphi(x,t) &= \exp(t)[x + \cos(\pi x) + \pi^2 \cos(\pi x)] - \\ &\exp(t)(1+t^2)[x + \cos(\pi x)], \end{aligned} \quad \dots(46)$$

$$g_0(0, t) = \exp(t), \quad \dots (47)$$

$$g_1(1, t) = 0, \quad \dots (48)$$

$$f(x) = \cos(\pi x) + x, \quad \dots(49)$$

$$k(x) = 1 + x^2 \quad \dots(50)$$

for which the exact solution is

$$u(x,t) = \exp(t)[\cos(\pi x) + x], \quad \dots(51)$$

and

$$p(t) = 1 + t^2. \quad (52)$$

Absolute error of $u(x,t)$ from example 1.

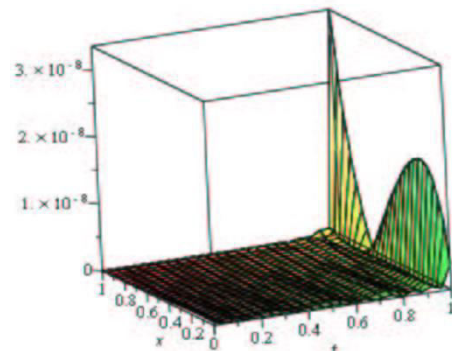


fig. 1: Estimated and exact solutions of $u(x,t)$ for example 1, with $m=n=11$.

Absolute error of $p(t)$ from example 1.

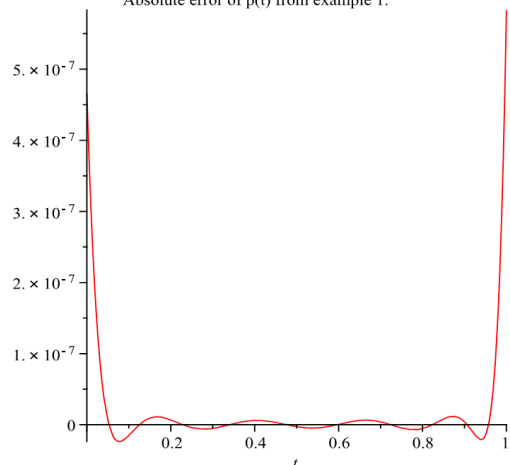


fig. 2: Estimated and exact solutions of $p(t)$ for example 1, with $m=n=11$.

Table 1: Absolute values of error for $u(x,1)$ from Example 1.

x	Crank-Nikolson [16]	$m=n=9$	$m=n=11$
0.1	6.8×10^{-3}	-0.2119×10^{-6}	1.399310×10^{-8}
0.2	6.9×10^{-3}	-0.2791×10^{-6}	1.7395×10^{-8}
0.3	6.7×10^{-3}	-0.2796×10^{-6}	1.6907×10^{-8}
0.4	6.5×10^{-3}	-0.2382×10^{-6}	1.4226×10^{-8}
0.5	6.7×10^{-3}	-0.1713×10^{-6}	1.02203×10^{-8}
0.6	6.7×10^{-3}	-8.7716×10^{-7}	5.5008×10^{-9}
0.7	6.9×10^{-3}	1.0688×10^{-7}	1.9651×10^{-10}
0.8	6.3×10^{-3}	0.130904×10^{-6}	-6.4844×10^{-9}
0.9	6.7×10^{-3}	0.2943×10^{-6}	-1.6395×10^{-8}
1	6.2×10^{-3}	0.5426×10^{-6}	-3.3585×10^{-8}

Table 2: Absolute values of error for $p(t)$ from Example 1.

x	Crank-Nikolson [16]	$m=n=11$
0.1	7.4×10^{-3}	-1.5082×10^{-8}
0.2	7.7×10^{-3}	6.7122×10^{-9}
0.3	7.6×10^{-3}	-5.2104×10^{-9}
0.4	7.9×10^{-3}	6.0173×10^{-9}
0.5	7.7×10^{-3}	-2.758×10^{-9}
0.6	7.7×10^{-3}	5.1636×10^{-10}
0.7	7.6×10^{-3}	4.0369×10^{-9}
0.8	7.3×10^{-3}	-5.5089×10^{-9}
0.9	7.3×10^{-3}	4.13982×10^{-9}
1	7.1×10^{-3}	5.8328×10^{-7}

Example 2. In this example we will solve the problem (30-34) with the given data (Table 3, 4 and Fig 3, 4)

$$\varphi(x, t) = (\pi^2 - (t+1)^2) \exp(-t^2) (\cos(\pi x) + \sin(\pi x)), \quad \dots (53)$$

$$g_0(t) = \exp(-t^2), \quad \dots (54)$$

$$g_1(t) = -\exp(-t^2), \quad \dots (55)$$

$$E(t) = \sqrt{2} \exp(-t^2), \quad \dots (56)$$

$$f(x) = \cos(\pi x) + \sin(\pi x), \quad \dots (57)$$

Whose exact solutions is

$$u(x, t) = \exp(-t^2) (\cos(\pi x) + \sin(\pi x)), \quad \dots (58)$$

$$p(t) = 1 + t^2, \quad \dots (59)$$

Absolute error of $u(x,t)$ from example 2.

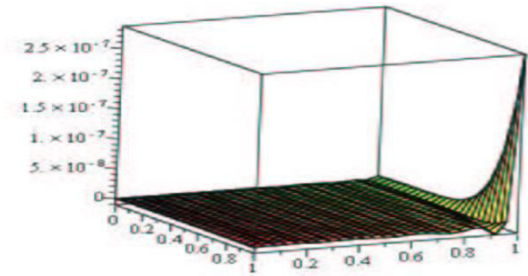


fig. 3: Estimated and exact solutions of $u(x,t)$ for example 2, with $m=n=11$.

Absolute error of $p(t)$ from example 2.

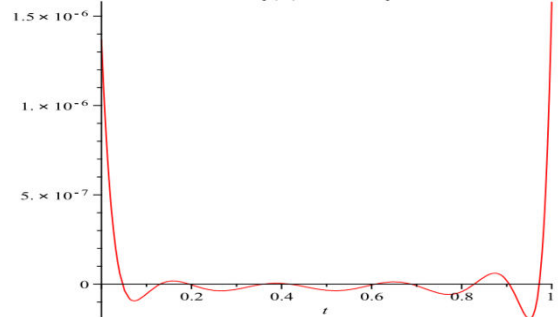


fig. 4: Estimated and exact solutions of $p(t)$ for example 1, with $m=n=11$.

Table 3: Absolute values of error for $u(x,1)$ from Example 2.

x	Crank-Nikolson [16]	$m=n=9$	$m=n=11$
0.1	3.3×10^{-3}	-3.7274×10^{-7}	5.8706×10^{-9}
0.2	3.3×10^{-3}	-3.9770×10^{-7}	6.3719×10^{-9}
0.3	3.2×10^{-3}	-4.3991×10^{-7}	6.4056×10^{-9}
0.4	3.5×10^{-3}	-6.0587×10^{-7}	6.6705×10^{-9}
0.5	3.6×10^{-3}	-0.1043×10^{-6}	8.0970×10^{-9}
0.6	3.7×10^{-3}	-0.1960×10^{-6}	1.3646×10^{-8}
0.7	3.8×10^{-3}	-0.3868×10^{-6}	2.8332×10^{-8}
0.8	3.9×10^{-3}	-0.7558×10^{-9}	6.2849×10^{-8}
0.9	3.8×10^{-3}	-0.1434×10^{-5}	1.3684×10^{-7}
1	3.7×10^{-3}	-0.26465×10^{-5}	2.86805×10^{-7}

Table 4: Absolute values of error for $p(t)$ from Example 2.

x	Crank-Nikolson [16]	$m=n=11$
0.1	5.2×10^{-3}	-5.4066×10^{-8}
0.2	5.0×10^{-3}	-3.5906×10^{-9}
0.3	5.4×10^{-3}	-2.8622×10^{-8}
0.4	5.6×10^{-3}	5.1109×10^{-9}
0.5	5.6×10^{-3}	-3.4208×10^{-8}
0.6	5.2×10^{-3}	-3.9743×10^{-9}
0.7	5.1×10^{-3}	-8.3392×10^{-9}

0.8	5.0×10^{-3}	-4.1893×10^{-8}
0.9	5.3×10^{-3}	2.1311×10^{-8}
1	5.3×10^{-3}	0.1580×10^{-6}

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5. CONCLUSIONS

The properties of the Bernstein polynomials are used to reduce the parabolic Inverse problem Subject to Temperature Overspecification semilinear parabolic partial differential equations with nonclassic boundary specifications to the solution of system of equations. From the solutions obtained using the suggested method we can conclude that these solutions are in excellent agreement with the already existing one.([16]).

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