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Wave Equation at Initial Conditions, Fourier Series Solution and Application to EEG Curves

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***Abstract* – In this article, the author make comparison between the soltion of the 3d-wave equation and the 1d-wave equation. The solution is unique when the equation has an initial and boundry condition. Also, the solution of the eqution can be written on the form of Fourier series. A program on Scilab is written. It present the spectrum of the series of Fourier and the solution curves. The author present an applicantion on the EEG curves as a solution of a wave equation.**

***Keywords* – Wave Equation, Fourier Series, EEG, Scilab, Fourier Spectrum, Fourier Curves.**

I. Introduction

In the article [3], the author shows that the Maxwell system of equations can be transformed into a couple of wave equations. Therefore, the problem of identifying the electric field E is reduced to solving the d'Alembert equation. A particular case of the dimension 1, has been treated in the article [**7**]: it is the general form of a solution of the Homogeneous Equation in one Dimension.

Theorem 1

We have, u is a solution of,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

If and only if there are two twice derivable functions f and g such that,

$$u(x,t) = f(x-ct) + g(x+ct) \tag{2}$$

In the article [3], the author recalls the definition of partial differential equations of the second hyperbolic order, then she is interested in solving the wave equation in dimension 3 via the following theorem.

Theorem 2

Let be the parallelogram R =] 0, a[×] 0, b[×] 0, c [and let be the following problem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} & t > 0 \; and \; (x,y,z) \in R \\ u(x,y,z,0) = f(x,y,z) & (x,y,z) \in R \\ \frac{\partial u}{\partial t} = g(x,y,z) & (x,y,z) \in R \\ u = 0 & on \; the \; faces \; of \; R \end{cases} \tag{3}$$

It is assumed that the functions f and g are zero on the faces of R. A solution of the system ?? is:

$$u(x,y,z,t) = \sum_{l,m,n\in\mathbb{N}^*} (K_{l,m,n}cos(\delta t) + L_{l,m,n}sin(\delta t)) \sin\left(\frac{l\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \sin\left(\frac{n\pi}{c} z\right) \tag{4}$$

This is a Fourier series. By definition, the expressions of its coefficients are:

$$\begin{cases} K_{l,m,n} = \frac{8}{abc} \int_0^a \int_0^b \int_0^c f(x,y,z) \sin\left(\frac{l\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \sin(\frac{n\pi}{c} z)\, dxdydz \\ L_{l,m,n} = \frac{8}{abc\delta} \int_0^a \int_0^b \int_0^c g(x,y,z) \sin\left(\frac{l\pi}{a} x\right) \sin\left(\frac{m\pi}{b} y\right) \sin(\frac{n\pi}{c} z)\, dxdydz \end{cases} \quad (5)$$

with $\delta = \pi\sqrt{\frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}}$.

II. Mathematical Tools

We denote *f*(*t*) a signal of a wave in propagation as a function of time *t*. We say that *f* is periodic with period *T* if:

$$f(t + T) = f(t), \forall t \in R, with\, T > 0 \quad (6)$$

All the data of the signal are found in a curve of duration *T*. The number of data of length *T* that we find in an interval of duration one second, is called the frequency. In physics, frequency is the number of times a periodic phenomenon recurs per unit of time. Its unit in the S.I. is the Hertz (Hz), an international unit of measurement according to which 1 hertz is equal to one cycle per second.

$$\nu = \frac{1}{T} \quad (7)$$

The curve of the signal as a function of time has characteristics that can be measured as soon as the signal is converted into an electrical signal.

III. Relationship between the Solution 2 and the Solution 4

Let *u*(*x*, *t*) be a solution of the wave equation **(1)** that we recall:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad (8)$$

As mentioned in the formula (2), the solutions are written as a combination of two progressive waves:

$$u(x,t) = f\,(x - ct) + g(x + ct) \quad (9)$$

or again:

$$u(x,t) = \psi_1\,(t - \frac{x}{c}) + \psi_2\,(t + \frac{x}{c}) \quad (10)$$

To have a unique solution, it is necessary to add initial conditions. Some are temporal conditions others are spatial. We will take into consideration the two types of initial conditions for the construction of a unique solution in the form of a Fourier series.

Initial Spatial Conditions

Suppose that the waves are stationary on a segment [0, *L*]. For example, this is the case of a string of length *L* stretched at its ends *A*(0) and *B*(*L*) and that we pinch. Since there is no wave vibration at the ends, the spatial boundary conditions are:

$$\begin{cases} u(0,t) = 0 \;, \forall t \in [0,T]. \\ u(L,t) = 0 \;, \forall t \in [0,T]. \end{cases} \quad (11)$$

Using the formula (10), we find

$$\begin{cases} u(0,t) = \psi_1(t) + \psi_2(t), & ,\forall t \in [0,T] \\ u(L,t) = \psi_1\left(t - \frac{L}{c}\right) + \psi_2\left(t + \frac{L}{c}\right), & \forall t \in [0,T] \end{cases} \tag{12}$$

That is to say:

$$\begin{cases} \psi_1(t) = -\psi_2(t) & ,\forall t \in [0,T] \\ \psi_1\left(t - \frac{L}{c}\right) = -\psi_2(t + \frac{L}{c}) & ,\forall t \in [0,T] \end{cases} \tag{13}$$

Therefore, the second relation becomes

$$\psi_1\left(t - \frac{L}{c}\right) = \psi_1(t + \frac{L}{c}) \tag{14}$$

In other words, ψ_1 is a periodic function with period $T = \frac{2L}{c}$ and therefore frequency $\nu_0 = \frac{c}{2L}$. Thus, according to Fourier's theorem, ψ_1 can decompose into a Fourier series as follows:

$$\psi_1(t) = \sum_{n\in\mathbb{N}^*} a_n cos(\delta t) + b_n sin(\delta t) \tag{15}$$

where $\delta = 2\pi n\nu_0$. By injecting this relation into 10, we obtain

$$u(x,t) = \sum_{n\in\mathbb{N}^*}(a_n \cos(\delta t) + b_n \sin(\delta t)) \sin(\frac{n\pi}{L} x) \tag{16}$$

Initial Temporal Conditions

Moreover, the initial temporal conditions make it possible to calculate the Fourier coefficients a_n and b_n in the expression 16. Indeed, these conditions are:

$$\begin{cases} u(x,0) = f(x), x \in [0,L] \\ \frac{\partial u}{\partial t}(x,0) = g(x), x \in [0,L] \end{cases} \tag{17}$$

Using the expression in the form of a Fourier series, the conditions for temporal imitations are written:

$$\begin{cases} f(x) = \sum_{n\in\mathbb{N}^*} a_n \sin(\frac{n\pi}{L} x), & x \in [0,L] \\ g(x) = \sum_{n\in\mathbb{N}^*} \delta\, b_n \sin\left(\frac{n\pi}{L} x\right), & x \in [0,L] \end{cases} \tag{18}$$

The first relation makes it possible to interpret a_n as a Fourier series of a periodic signal of period $T = 2L$. By definition of the integral formulas of the coefficients of the Fourier series:

$$\begin{cases} a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right) dx \\ b_n = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L} x\right) dx \end{cases} \tag{19}$$

with $\delta = 2\pi n\nu_0$.

IV. Program in Scilab

The program below calculates the Fourier coefficients from integral formulas and illustrates the temporal and spectral curves. The software can be downloaded for free at https://www.scilab.org/ fr

3.1. *The Scilab Program*

```
clf()
```

```
deff('z = f(x, t)',' z = cos(t)+x+1')

// computes the integral over the square [0,1]×[0,1]

[I0, e] = int2d(0, 1,0,1, f)

disp(tlist('[I,e] =',I0,e))

//coefficient a0

a0 = 0.5 ∗ L ∗ I0;

plot2d(0,a0,-2); gcf() ;

//coefficients an

for n = 1 : 5

deff('zn = fn(x,t)',' zn = (cos(t)+x+1)∗ sin(n∗Lπ ∗ x)')

an = (1/L)∗ int2d(0,1,0,1, fn)

plot2d(n,an,-1);

gcf()

end

//coefficients bm

for m = 1 :5

deff('zm=fm(x,t)','zm=(cos(t)+x+1)*sin(n*π L∗x)')

bm = (1/L)∗ int2d(0,1,0,1, f m)

plot2d(m,bm,-5)

legends(['a0 = b0''an'bn'],[-2,-1,-5],1)

title("Fourier series spectrum"); end

//Time function with x=0.5 scf(5);

t = 0 : 20;

plot(t,cos(t)+1.5);

legend([' f(t) = cos(t)+1.5;x = 0.5'])

title("Time curve between 0 and 20 seconds at the spatial position x=0.5,");
```

3.2. *The Results Obtained*

In the program, the function f considered is, $f(x,t) = cos(t) + x + 1$. At the spatial position $= 0.5$, the time curve between 0 and 20 seconds is presented in the figure 1, like in the standard curves made by the electroencephalograms software. In this case, the function becomes, $f(0.5,t) = cos(t) + 1.5$

The result illustrated in the figures 2 is the fourier series spectrum. It corresponds to the histogram of the four-

-ier coefficients a_n and b_n for $n = 0$ to 5.

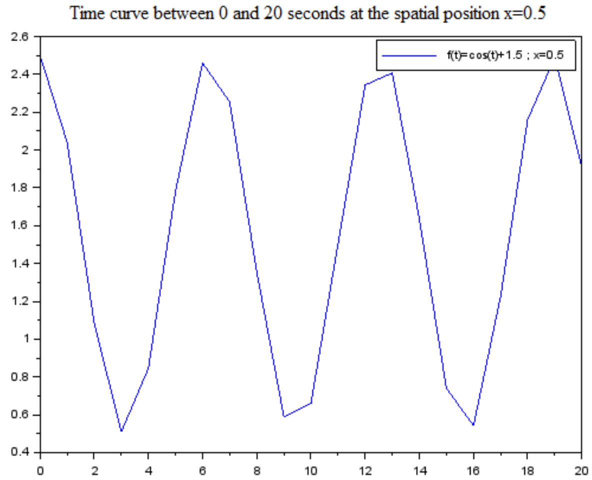

Fig. 1. Temporal representation for fixed spatial position.

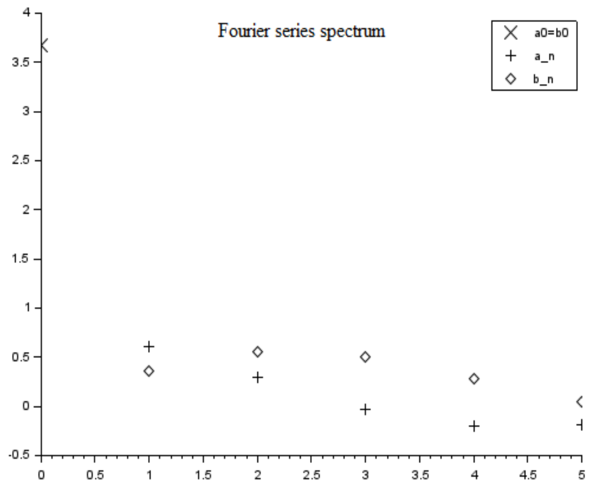

Fig. 2. Spectral representation (n, an, bn).

V. Application: Reading an EEG graph

The outlet of an EEG graph contains 20 seconds of recordings of the data captured by the 21 electrodes of the EEG headset and an ECG signal sensor: electrocardiogram activity. In addition, if an SLI light stimulation test is carried out then the detected signals are recorded at the bottom of the sheet. Sometimes the result of an EEG exam consists of 20 curves depending on the time. Therefore, the first 18 curves represent the curves of the brain waves detected by the 18 sensors of the EEG headset distributed according to the lobes of the brain. The 19*th* curve represents the vibrational activity detected by the electrocardiogram. And the last line represents the brain wave curve due to eye movements during an SLI test.

The tracing is carried out in a Cartesian reference frame, the x-axis represents the time in seconds, it is graduated up to 20 second. On the other hand, the ordinate axis represents the values of the function $u_i(t)$, with i the index of the electrod of the EEG headset. Each model function u_i models a brain wave relating to the activity of a lobe. Generally, the sensors are separated by letters and numbers as follows: The letters indicate the precise location of the scalp: FP: fronto-polar; F: frontal; T: temporal; C: central; P: parietal; O: occipital. Even numbers mean the right side of the scalp, and the odd number means the left side of the scalp.

The tracing plane is toast. Each grid is 1 second long. The number of variations of a curve in 1 second represents the frequency of the brain wave. Which allows its classification. The curves represented are functions of time $f(t)$, to classify the belonging of a wave, it suffices to detect the frequency, that is to say the number of periodic graphs per second. In practice, to classify a wave on an EEG plot, it suffices to count the number of repetitions of a curve in one second. In the article [2], the author presented a classification of brain waves according to their frequency:

1) Delta waves: from 0.5 to 4 Hz, those of deep sleep.
2) Theta waves: from 4 to 7 Hz, those of deep relaxation, in full awakening, reached especially by experienced meditators.
3) Alpha waves: from 8 to 13 Hz, those of light relaxation and calm awakening.
4) Beta waves: from 14 Hz to 29 Hz, those of current activities.
5) Gamma waves: from 30 to 100 Hz, those of a working memory.

In conclusion, the EEG plots are classified according to their curves (number of repetitions in one second) and not on the overall shape of the curve, without taking into account the content of the wave signals. The reading of an EEG graph also depends on the sensors and their locations. For example, the frontal lobe is that of motor skills. The parietal lobe is mainly sensitive with projection areas of tactile, thermal, proprioceptive sensisibility. Some areas are related to language. Others to writing. The temporal lobe contains the center of Wernick’s language. Further forward, there are the auditory projection areas or sound interpretation areas. Further inside, there are the olfactory areas. He is the main center of intelligence. Finally, the center of visual perception is located in the occipital lobe, so visual activity is plotted on the graph corresponding to sensors O, $O1$ and $O2$.

VI. Application: Reading an EEG Graph

In this article, the author uses results dating from the 18th and 19th centuries to present a solution to a recent problem: the modeling of brain waves via the wave equation. However, it is difficult to model brain activity with precision while certain constants are missing such as the permeability and the permittivity of the brain tissue are not quantitatively measured.

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AUTHOR'S PROFILE

A. Yosra Annabi, has a background in applied mathematics. She works on EDP, direct and inverse mathematical problems, mathematical modeling in medicine, hydrogeology and emotion.