A Decomposition Algorithm for Solving a Special Type of Two-Level Integer Linear Multiple Objectives Decision Making Problems Using TOPSIS

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Abstract: This paper extended TOPSIS (Technique for Order Preference by Similarity Ideal Solution) for solving a special type of Two-Level Integer Linear Multiple Objectives Decision Making (ST-TL-IL MODM) Problems with block angular structure. In order to obtain a compromise (satisfactory) integer solution to the ST-TL-IL MODM problems with block angular structure using the proposed TOPSIS approach, a modified formulas for the distance function from the positive ideal solution (PIS) and the distance function from the negative ideal solution (NIS) are proposed and modeled to include all the objective functions of the two levels. In every level, as the measure of “Closeness” $d_i$-metric is used, a k-dimensional objective space is reduced to two-dimensional objective space by a first-order compromise procedure. The membership functions of fuzzy set theory is used to represent the satisfaction level for both criteria. A single-objective programming problem is obtained by using the max-min operator for the second – order compromise operation. A decomposition algorithm for generating a compromise (satisfactory) integer solution through TOPSIS approach is provided where the first level decision maker (FLDM) is asked to specify the relative importance of the objectives. Finally, an illustrative numerical example is given to clarify the main results developed in the paper.

Keywords: Two-Level Integer Multiple Objectives Decision Making Problems, TOPSIS, Decomposition Techniques, Fuzzy Sets.

I. INTRODUCTION

Two-level multiple objectives decision making problems consist of the objectives of the leader at its first level and that of the follower at the second level. The decision maker (DM) at each level attempts to optimize his individual objectives, which usually depend in part on the variables controlled by the DM at the other levels and their final decisions are executed sequentially where the first level decision maker (FLDM) makes his decision firstly. The research and applications concentrated mainly on two-level programming (see f. i. [7, 8, 11, 14, 15, 16, 18, 19, 20, 21, 22, 24, 25, 26, 34,35,36, 51,53,56,57]).

TOPSIS was first developed by C. L. Hwang and K. Yoon [40] for solving a multiple attributes decision making (MADM) problems. It is based upon the principle that the chosen alternative should have the shortest distance from the PIS and the farthest from the NIS. After the publication of TOPSIS method [40, 44], the subsequent works in this area of optimization have been numerous (see f. i. [1, 3,4,5,6,9,10,12,13, 17, 27, 28,29, 33, 41, 42, 44,54]).


Recently, M. A. Abo-Sinna and T. H. M. Abou-El-Enien [11] extend TOPSIS for solving Large Scale Bi-level Vector Optimization problems (LS-BL-LVOP); they further extended the concept of TOPSIS [Lia et al. (44)] for LS-BL-LVOP.

This paper extended TOPSIS for solving a ST-TL-IL MODM problems with block angular structure, also, the concept of TOPSIS is extended [Lia et al. (44)] for a ST-TL-IL MODM problems with block angular structure.

The following section will give the formulation of a ST-TL-IL MODM problems with block angular structure. The family of $d_i$-distance and its normalization is discussed in section 3. The TOPSIS approach for a ST-TL-IL MODM problems with block angular is presented in section 4. By use of TOPSIS, a decomposition algorithm is proposed for solving a ST-TL-IL MODM problems with block angular structure in section 5. An illustrative numerical example is given in section 6. Finally, summary and conclusions will be given in section 7.

II. FORMULATION OF A ST-TL-IL MODM PROBLEMS WITH BLOCK ANGULAR STRUCTURE

Consider the case when there are two levels in a hierarchy structure with a first level decision maker (FLDM) and a second level decision maker (SLDM). Let the ST-TL-IL MODM problem of the following block angular structure be:

**[FLDM]**

$\begin{align*}
\text{Maximize} & \quad Z_{i_1}(X_{i_1},X_{i_2}) = \\
\text{Subject to} & \quad z_{i_11}(X_{i_1},X_{i_2}), ..., z_{i_1k_1}(X_{i_1},X_{i_2}) \geq 0
\end{align*}$

where $X_{i_2}$ solves second level

**[SLDM]**
Maximize
\[ X_{l_2} \quad Z_{l_2}(X_{l_1}, X_{l_2}) = \]

Maximize
\[ X_{l_2} \quad (z_{l_2}(X_{l_1}, X_{l_2}), \ldots, z_{l_2}(X_{l_1}, X_{l_2})) \]
subject to \( \quad (1) \)
\[ X \in M = \{X \in R^n; \sum^n_j A_j x_j \leq b_o, \quad D_j X_j \leq b_j, \quad X_j \geq 0, \text{and integer, } j = 1, 2, \ldots, q, q > 1\} \]

where
\( k \) : the number of objective functions,
\( k_{l_1} \) : the number of objective functions of the FLDM,
\( k_{l_2} \) : the number of objective functions of the SLDM,
\( n_{l_1} \) : the number of variables of the FLDM,
\( n_{l_2} \) : the number of variables of the SLDM,
\( q \) : the number of subproblems,
\( m \) : the number of constraints,
\( n \) : the number of variables,
\( n_j \) : the number of variables of the \( j^{th} \) subproblem,
\( m_o \) : the number of the common constraints represented by \( \sum^n_j A_j x_j \leq b_o \)
\( m_j \) : the number of independent constraints of the \( j^{th} \) subproblem represented by \( D_j X_j \leq b_j, j = 1, 2, \ldots, q \).
\( A_j \) : an \( (m_o \times n_j) \) coefficient matrix,
\( D_j \) : an \( (m \times n_j) \) coefficient matrix,
\( b_j \) : an \( m \)-dimensional column vector of right-hand sides of the common constraints whose elements are constants,
\( b_j \) : an \( m \)-dimensional column vector of independent constraints right-hand sides whose elements are the constants of the constraints for the \( j^{th} \) subproblem,
\( C_0 \) : an \( n_j \)-dimensional row vector for the \( j^{th} \) subproblem in the \( \rho^o \) objective function,
\( R \) : the set of all real numbers,
\( X \) : an \( n \)-dimensional column vector of variables,
\( X_j \) : an \( n_j \)-dimensional column vector of variables for the \( j^{th} \) subproblem, \( j = 1, 2, \ldots, q \).

If the objective functions are linear, then the objective function can be written as follows:
\[ z_i(X) = \sum^q_j C_{ij} X_j, \quad i = 1, 2, \ldots, k \] (2)

### III. SOME BASIC CONCEPTS OF DISTANCE MEASURES

The compromise programming approach [37, 45, 58, 59] has been developed to perform multiple objectives decision making (MODM) problems, reducing the set of nondominated solutions. The compromise solutions are those which are the closest by some distance measure to the ideal one.

The point \( z_i(X^*) = \sum^q_j z_{ij}(X^*) \) in the criteria space is called the ideal point (reference point). As the measure of “closeness”, \( d_p \)-metric is used. The \( d_p \)-metric defines the distance between two points, \( z_i(X) = \sum^q_j z_{ij}(X) \) and \( z_i(X^*) = \sum^q_j z_{ij}(X^*) \) (the reference point) in \( k \)-dimensional space [49] as:

\[ d_p = \left( \sum^{k}_{i=1} w^p_i \left( z^*_i - z_i \right)^p \right)^{1/p} \]

where \( p \geq 1 \).

Unfortunately, because of the incommensurability among objectives, it is impossible to directly use the above distance family. To remove the effects of the incommensurability, we need to normalize the distance family of equation (3) by using the reference point [39, 40] as:

\[ d_p = \left( \sum^{k}_{i=1} w^p_i \left( \frac{\sum^q_j z^*_j - \sum^q_j z_{ij}}{\sum^q_j z^*_j} \right)^p \right)^{1/p} \]

where \( p \geq 1 \).

To obtain a compromise integer solution for the Integer large scale multiple objectives decision making (ILSMODM) problem of the following block angular structure:

Maximize \( [z_1(X), z_2(X), \ldots, z_n(X)] \)
subject to \( \quad (5) \)
\[ X \in M = \{X \in R^n; \sum^n_j A_j x_j \leq b_o, \quad D_j X_j \leq b_j, \quad X_j \geq 0, \text{and integer, } j = 1, 2, \ldots, q, q > 1\} \]

The global criteria method [39] for ILSMODM [2, 48] uses the distance family of equation (4) by the ideal solution being the reference point. The problem becomes how to solve the following auxiliary problem:

\[ \min_{x \in M} d_p = \left( \sum^{k}_{i=1} w^p_i \left( \frac{\sum^q_j z_{ij}(X^*) - \sum^q_j z_{ij}(X)}{\sum^q_j z^*_j(X^*)} \right)^p \right)^{1/p} \]

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where \( X^* \) is the PIS and \( p = 1, 2, \ldots, \infty \).

Usually, the solutions based on PIS are different from the solutions based on NIS. Thus, both PIS(\( z^* \)) and NIS(\( z^- \)) can be used to normalize the distance family and obtain [4]:

\[
d_p = \left( \sum_{i=1}^{k} w_i^p \left( \frac{\sum_{j=1}^{q} z_{ij} - \sum_{j=1}^{q} z_{ij}^*}{\sum_{j=1}^{q} z_{ij}^* - \sum_{j=1}^{q} z_{ij}} \right)^p \right)^{1/p}
\]

where \( p \geq 1 \).

IV. TOPSIS FOR A ST-TL-IL MODM PROBLEMS WITH BLOCK ANGULAR STRUCTURE

Consider the following ST-TL-IL MODM problem with block angular structure:

[FLDM]

\[
\begin{align*}
\text{Maximize/Minimize} & \quad Z_{i1} (X_{i1}, X_{i2}) = \\
\text{Maximize/Minimize} & \quad (z_{i11}(X_{i1}, X_{i2}), \ldots, z_{i1k_{i1}}(X_{i1}, X_{i2}))
\end{align*}
\]

subject to

\[
X \in M
\]

where \( \sum_{j=1}^{q} z_{ij} (X) \): Objective Function for Maximization, \( t \in K_1 \subseteq K \),
\( \sum_{j=1}^{q} z_{ij} (X) \): Objective Function for Minimization, \( v \in K_2 \subseteq K \).

4-1. Phase (I)

Consider the FLDM problem of the ST-TL-IL MODM problem (8): [FLDM]

\[
\begin{align*}
\text{Maximize/Minimize} & \quad Z_{i1} (X_{i1}, X_{i2}) = \\
\text{Maximize/Minimize} & \quad (z_{i11}(X_{i1}, X_{i2}), \ldots, z_{i1k_{i1}}(X_{i1}, X_{i2}))
\end{align*}
\]

subject to

\[
X \in M
\]

where \( \sum_{j=1}^{q} z_{ij} (X) \): Objective Function for Maximization, \( t \in K_1 \subseteq K \),
\( \sum_{j=1}^{q} z_{ij} (X) \): Objective Function for Minimization, \( v \in K_2 \subseteq K \).

In order to use the distance family of equation (7) to resolve problem (9), we must first find PIS(\( z^* \)) and NIS(\( z^- \)) which are [4, 44]:

\[
\begin{align*}
Z_{FLDM}^p & = \max \left\{ \frac{q}{\sum_{j=1}^{q} x_{ij}^F (X)} \right\} \quad \forall t (and \; v) \\
Z_{FLDM}^p & = \min \left\{ \frac{q}{\sum_{j=1}^{q} x_{ij}^F (X)} \right\} \quad \forall t (and \; v)
\end{align*}
\]

where \( K = K_1 \cup K_2 \).

Using the PIS and the NIS for the FLDM, we obtain the following distance functions from them, respectively:

\[
\begin{align*}
d_p^{PFISM} & = \left( \sum_{i \in K_1} w_i^{-p} \left( \frac{q}{\sum_{j=1}^{q} x_{ij}^F (X)} - \frac{q}{\sum_{j=1}^{q} x_{ij}^F (X)} \right) \right)^{1/p} \\
d_p^{PFISM} & = \left( \sum_{i \in K_2} w_i^{-p} \left( \frac{q}{\sum_{j=1}^{q} x_{ij}^F (X)} - \frac{q}{\sum_{j=1}^{q} x_{ij}^F (X)} \right) \right)^{1/p}
\end{align*}
\]

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represent these individual optima. Assume that the membership functions \((\mu_1(X) \text{ and } \mu_2(X))\) of two objective functions are linear. Then, based on the preference concept, we assign a larger degree to the one with shorter distance from the PIS for \(\mu_1(X)\) and assign a larger degree to the one with farther distance from NIS for \(\mu_2(X)\). Therefore, as shown in figure (1), \(\mu_1(X) \equiv \mu_{dp^{FLDM}}(X)\) and \(\mu_2(X) \equiv \mu_{dp^{NIS^{FLDM}}}(X)\) can be obtained as the following (see [23, 42, 43, 52, 51, 60]):

\[
\mu_1(X) = \begin{cases} 
1, & \text{if } d_{p^{NIS^{FLDM}}}(X) < (d_{p^{NIS^{FLDM}}})^*, \\
1 - \left(\frac{d_{p^{NIS^{FLDM}}}(X) - d_{p^{NIS^{FLDM}}}}{d_{p^{NIS^{FLDM}}}}\right)^*, & \text{if } (d_{p^{NIS^{FLDM}}})^* \geq d_{p^{NIS^{FLDM}}}(X) \geq (d_{p^{NIS^{FLDM}}})^*, \\
0, & \text{if } d_{p^{NIS^{FLDM}}}(X) > (d_{p^{NIS^{FLDM}}})^*, 
\end{cases}
\]

(13-a)

\[
\mu_2(X) = \begin{cases} 
1, & \text{if } d_{p^{NIS^{FLDM}}}(X) > (d_{p^{NIS^{FLDM}}})^*, \\
1 - \left(\frac{d_{p^{NIS^{FLDM}}}(X) - d_{p^{NIS^{FLDM}}}}{d_{p^{NIS^{FLDM}}}}\right)^*, & \text{if } (d_{p^{NIS^{FLDM}}})^* \leq d_{p^{NIS^{FLDM}}}(X) \leq (d_{p^{NIS^{FLDM}}})^*, \\
0, & \text{if } d_{p^{NIS^{FLDM}}}(X) < (d_{p^{NIS^{FLDM}}})^*, 
\end{cases}
\]

(13-b)

\[\begin{align*}
&\text{Figure (1): The membership functions of } \\
&\mu_{dp^{NIS^{FLDM}}}(X) \text{ and } \mu_{dp^{PIS^{FLDM}}}(X)
\end{align*}\]

where

\[
\begin{align*}
(d_{p^{NIS^{FLDM}}})^* &= \min_{x \in M} d_{p^{NIS^{FLDM}}}(X) \text{ and the solution is } X_{p^{NIS^{FLDM}}}, \\
(d_{p^{PIS^{FLDM}}})^* &= \max_{x \in M} d_{p^{PIS^{FLDM}}}(X) \text{ and the solution is } X_{p^{PIS^{FLDM}}}, \\
(d_{p^{NIS^{FLDM}}})^- &= d_{p^{NIS^{FLDM}}}(X_{p^{NIS^{FLDM}}}) \text{ and } \\
(d_{p^{PIS^{FLDM}}})^- &= d_{p^{PIS^{FLDM}}}(X_{p^{PIS^{FLDM}}}).
\end{align*}
\]

Now, by applying the max-min decision model which is proposed by R. E. Bellman and L. A. Zadeh [23] and extended by H. – J. Zimmermann [60], we can resolve problem (12). The satisfying decision of the FLDM of the LS-BL-LVOP Problem, \(X^{FLDM} = (X_{f_1}^{FLDM}, X_{f_2}^{FLDM})\), may be obtained by solving the following model:

\[
\mu_p(X^{FLDM}) = \max_{x \in M} \left\{ \min \left( \mu_1(X), \mu_2(X) \right) \right\} \quad (14)
\]

Finally, if \(\delta^{FLDM} = \min \left( \mu_1(X), \mu_2(X) \right)\), the model (14) is equivalent to the form of Tchebycheff model (see [32]), which is equivalent to the following model:

\[
\max \gamma^{FLDM},
\]

subject to

\[
\mu_1(X) \geq \gamma^{FLDM}, \quad (15-b)
\]

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where $\delta^{FLDM}$ is the satisfactory level for both criteria of the shortest distance from the PIS and the farthest distance from the NIS. It is well known that if the optimal solution of (15) is the vector $(y^{FLDM}, x^{FLDM})$, then $x^{FLDM}$ is a nondominated solution [39, 55] of (12) and a satisfactory solution [45] of the FLDM problem (9).

The basic concept of the bi-level programming technique is that the FLDM sets his/her goals and/or decisions with possible tolerances which are described by membership functions (Figure 2) for each of the $n_1$ components of the decision vector $x^{FLDM}$. Following Pramanik & Roy [50] and Sinha [53], this shift can be achieved.

4.2. Phase (II)  

The SLDM problem can be written as follows: [SLDM]

\[
\begin{align*}
\text{Maximize} & \quad Z_{ij}(x_i, x_j) = x_{ij} \\
\text{Maximize} & \quad \left( z_{i21}(x_{i1}, x_{i2}), \ldots, z_{i2n}(x_{i1}, x_{i2}) \right) \\
\text{subject to} & \quad \sum_{j=1}^{q} z_{ij}(x) = \text{Objective Function for Maximization,} \\
& \quad \sum_{j=1}^{q} z_{ij}(x) = \text{Objective Function for Minimization,}
\end{align*}
\]  

(17)

Figure (2): The membership function of the decision variable $x_{ij}$
Maximize or Minimize $\sum_{j=1}^{q} z_{ij}(x)$ (or $\sum_{j=1}^{q} z_{ij}(x)$).

∀(and v)

∀(and v)

where $K = K_1 \cup K_2$.

$z_{\text{SLDM}} = (z_{1}^{\text{SLDM}}, z_{2}^{\text{SLDM}}, \ldots, z_{k_2}^{\text{SLDM}})$ and $z_{\text{SLDM}} = (z_{1}^{\text{SLDM}}, z_{2}^{\text{SLDM}}, \ldots, z_{k_2}^{\text{SLDM}})$ are the individual positive (negative) ideal solutions for the SLDM.

In order to obtain a compromise (satisfactory ) integer solution to the ST-TL-II MODM problem using TOPSIS approach, the distance family of (7) to represent the distance function from the positive ideal solution, $d_{\text{PIS}}^p$, and the distance function from the negative ideal solution, $d_{\text{NIS}}^p$, can be proposed, in this paper, for the objectives of the FLDM and the SLDM as follows:

$d_{\text{PIS}}^p = \left( \sum_{i \in K_1} W_i^p \left( \frac{\mu_{\text{FLDM}}^i(z_{ij}(x)) - \mu_{\text{FLDM}}^j(z_{ij}(x))}{\mu_{\text{FLDM}}^j(z_{ij}(x)) - \mu_{\text{FLDM}}^j(z_{ij}(x))} \right)^p \right)^{1/p}$

$d_{\text{NIS}}^p = \left( \sum_{i \in K_2} W_i^p \left( \frac{\mu_{\text{SLDM}}^i(z_{ij}(x)) - \mu_{\text{SLDM}}^j(z_{ij}(x))}{\mu_{\text{SLDM}}^j(z_{ij}(x)) - \mu_{\text{SLDM}}^j(z_{ij}(x))} \right)^p \right)^{1/p}$

where $w_j = 1,2,\ldots,k$, are the relative importance (weights) of objectives, and $p = 1,2,\ldots,\infty$.

In order to obtain a compromise solution, we transfer problem (8) into the following bi-objective problem with two commensurable (but often conflicting) objectives [4, 44]:

Minimize $d_{\text{PIS}}^p(X)$

Maximize $d_{\text{NIS}}^p(X)$ subject to

$X \in M$

where $p = 1,2,\ldots,\infty$.

Since these two objectives are usually conflicting to each other, we can simultaneously obtain their individual optima. Thus, we can use membership functions to represent these individual optima. Assume that the membership functions ($\mu_3(X)$ and $\mu_4(X)$) of two objective functions are linear. Then, based on the preference concept, we assign a larger degree to the one with shorter distance from the PIS for $\mu_3(X)$ and assign a larger degree to the one with farther distance from the NIS for $\mu_4(X)$. Therefore, as shown in figure (3), $\mu_3(X) \equiv \mu_{\text{PIS}}^p(X)$ and $\mu_4(X) \equiv \mu_{\text{NIS}}^p(X)$ can be obtained as the following (see [23, 42, 43, 52, 60]):
Where

\[
(d_p^{\text{PIS}})^* = \text{Minimize}_{x \in \mathbb{X}} d_p^{\text{PIS}}(X) \text{ and the solution is } X^{\text{PIS}},
\]

\[
(d_p^{\text{NIS}})^* = \text{Maximize}_{x \in \mathbb{X}} d_p^{\text{NIS}}(X) \text{ and the solution is } X^{\text{NIS}},
\]

\[
(d_p^{\text{PIS}}) = d_p^{\text{PIS}}(X^{\text{NIS}}) \text{and } (d_p^{\text{NIS}}) = d_p^{\text{NIS}}(X^{\text{PIS}}).
\]

Now, by applying the max-min decision model which is proposed by R. E. Bellman and L. A. Zadeh [23] and extended by H. –J. Zimmermann [60], we can resolve problem (20). The satisfactory integer solution of the ST-TL-ILMODM, \(x^{\text{TL}}\), may be obtained by solving the following model:

\[
\mu_p(x^{\text{TL}}) = \text{Maximize}_{x \in \mathbb{X}} \left\{ \text{Min.} \left( \mu_3(X) , \mu_4(X) \right) \right\} \quad (22)
\]

Finally, if \(d_p^{\text{TL}} = \text{Minimize} ( \mu_3(X) , \mu_4(X) )\), the model (22) is equivalent to the form of Tchebycheff model (see [32]), which is equivalent to the following model:

\[
\text{Maximize } \gamma^{\text{TL}},
\]

subject to

\[
\mu_3(X) \geq \gamma^{\text{TL}},
\]

\[
\mu_4(X) \geq \gamma^{\text{TL}},
\]

\[
\frac{x_{1i}^{\text{FLDM}} - x_{1i}^{\text{TL}}}{t_i} \geq \gamma^{\text{TL}}, i = 1, 2, ..., n_{t1} \quad (23-d)
\]

\[
\frac{x_{1i}^{\text{FLDM}} - x_{1i}^{\text{TL}}}{t_i} \geq \gamma^{\text{TL}}, i = 1, 2, ..., n_{t1} \quad (23-e)
\]

\(x \in M, \gamma^{\text{TL}} \in [0,1], \quad (23-f)\)

where \(\delta^{\text{TL}}\) is the satisfactory level for both criteria of the shortest distance from the PIS and the farthest distance from the NIS. It is well known that if the optimal integer solution of (23) is the vector \((\gamma^{\text{TL}}, x^{\text{TL}})\), then \(x^{\text{TL}}\) is a nondominated solution of (20) and a satisfactory integer solution for the ST-TL-IL MODM problem with block angular structure [48].

V. A DECOMPOSITION ALGORITHM OF TOPSIS FOR SOLVING A ST-TL-IL MODM PROBLEMS

Thus, we can introduce the following decomposition algorithm to generate a set of satisfactory integer solutions for the ST-TL-IL MODM problems with block angular structure:

**The algorithm (Alg-I):**

**Phase (I):**

*Step 1.* Ignore the integer requirement in problem (9).

*Step 2.* Construct the PIS payoff table of problem (9) by using the decomposition algorithm [30, 31, 46], and obtain \(z^{\text{FLDM}} = (z_1^{\text{FLDM}}, z_2^{\text{FLDM}}, ..., z_k^{\text{FLDM}})\) the individual positive ideal solutions.

*Step 3.* Construct the NIS payoff table of problem (9) by using the decomposition algorithm, and obtain \(z^{\text{NISFLDM}} = (z_1^{\text{NISFLDM}}, z_2^{\text{NISFLDM}}, ..., z_k^{\text{NISFLDM}})\), the individual negative ideal solutions.

*Step 4.* Use equations (10 & 11) and the above steps (2 & 3) to construct \(d_p^{\text{PISFLDM}}\) and \(d_p^{\text{NISFLDM}}\).

*Step 5.* Transform problem (9) to the form of problem (12).

*Step 6.* (I) Ask the FLDM to select \( p = p^* \in \{1, 2, ..., \infty \} \), (II) Ask the FLDM to select \( w_i = w_i^* \), \( i = 1, 2, ..., k_{i1} \), where \( \sum_{i=1}^{k_{i1}} w_i = 1 \).

*Step 7.* Use steps (4 & 6) to compute \(d_p^{\text{PISFLDM}}\) and \(d_p^{\text{NISFLDM}}\).

*Step 8.* Construct the payoff table of problem (12): At \( p = 1 \).
use the decomposition algorithm [30, 31, 46].
At \( p \geq 2 \), use the generalized reduced gradient method, [46, 47], and obtain:
\[
d_p^{FLDM} = \left( (d_p^{1FLDM})^\tau, (d_p^{2FLDM})^\tau \right).
\]
\[
d_p^{NISFLDM} = \left( (d_p^{1NISFLDM})^\tau, (d_p^{2NISFLDM})^\tau \right).
\]
Step 9. Construct problem (15) by using the membership functions (13).
Step 10. Solve problem (15) to obtain \( (y^{FLDM}, X^{FLDM}) \).
Step 11. Ask the FLM to select the maximum negative and positive tolerance values \( \tau_i^+ \) and \( \tau_i^- \), \( i = 1, 2, \ldots, n_i \), on the decision vector \( X_i^{FLDM} = (x_{i1}, x_{i2}, \ldots, x_{in_i}) \).

**VI. AN ILLUSTRATIVE NUMERICAL EXAMPLE**

Consider the following ST-TL-IL MODM problem with block angular structure:

[FLDM]
Maximize
\[
x_1, x_2 \quad F_{11}(x) = 4x_1 + 6x_2 - x_3 + 7x_4 \quad (24 - 1)
\]

Maximize
\[
x_1, x_2 \quad F_{12}(x) = -2x_1 + 9x_2 + 13x_3 + x_4 \quad (24 - 2)
\]

Minimize
\[
x_1, x_2 \quad F_{13}(x) = -x_1 + 3x_2 - x_3 + x_4 \quad (24 - 3)
\]

Minimize
\[
x_1, x_2 \quad F_{14}(x) = 6x_1 - 2x_2 + x_3 + x_4 \quad (24 - 4)
\]

where \( x_1, x_2 \) solves the second level

[SLDM]
Maximize
\[
x_3, x_4 \quad F_{21}(x) = x_1 - 5x_2 + 3x_3 + 19x_4 \quad (24 - 5)
\]

Minimize
\[
x_3, x_4 \quad F_{22}(x) = 4x_1 + x_2 + x_3 - x_4 \quad (24 - 6)
\]

subject to
\[
x_1 + x_2 + x_3 + x_4 \leq 66, \quad (24 - 7)
5x_1 + x_2 \leq 13, \quad (24 - 8)
x_3 + x_4 \leq 50, \quad (24 - 9)
x_1, x_2, x_3, x_4 \geq 0 \ and \ integer \ (24-10)
\]

Solution:
- Ignore the integer requirement in problem (24).
- Obtain PIS and NIS payoff tables for the FLM of problem (24).

**Table (1) : PIS payoff table for the FLM of problem (24)**

<table>
<thead>
<tr>
<th></th>
<th>( f_{11}(x) )</th>
<th>( f_{12}(x) )</th>
<th>( f_{13}(x) )</th>
<th>( f_{14}(x) )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximize ( x_1, x_2 )</td>
<td>( 428^* )</td>
<td>167</td>
<td>89</td>
<td>87</td>
<td>0</td>
<td>13</td>
<td>0</td>
<td>50</td>
</tr>
<tr>
<td>Maximize ( x_1, x_2 )</td>
<td>28</td>
<td>767*</td>
<td>-11</td>
<td>37</td>
<td>0</td>
<td>13</td>
<td>50</td>
<td>0</td>
</tr>
</tbody>
</table>

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Minimize & $f_{13}(X)$ & -39.6 & 644.8 & -52.6 & 65.6 & 2.6 & 0 \hline
Minimize & $f_{14}(X)$ & 78 & 117 & 39 & -13 & 0 & 13 \hline
\end{tabular}
\end{table}

$PIS: f^{FLDM} = (428, 767, -52.6, -13)$

| Table (2): NIS payoff table for the FLDM of problem (24) |
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Minimize $f_{11}(X)$ & $f_{11}$ & $f_{12}$ & $f_{13}$ & $f_{14}$ & $x_1$ & $x_2$ & $x_3$ & $x_4$ \\
\hline
$\begin{align*}
Minimize f_{11}(X) && -50.0 & 650 & -50 & 50 & 0 & 0 & 50 & 0 \\
Minimize f_{12}(X) && 10.4 & -5.2 & -2.6 & 15.6 & 2.6 & 0 & 0 & 0 \\
Maximize f_{13}(X) && 428 & 167 & 89 & 87 & 0 & 13 & 0 & 50 \\
Maximize f_{14}(X) && 360.4 & 44.8 & 47.4 & 115.6 & 2.6 & 0 & 0 & 50 \\
\end{align*}$
\end{tabular}

$NIS: f^{-}_{FLDM} = (-50, -5.2, 89, 115.6)$

- Next, compute equation (11) and obtain the following equations:

\[
d^{FLDM}_{P}\text{NIS} = \left[w_{11}^{P} \left(\frac{f_{11}(X) - (-50)}{428 - (-50)}\right)^{p} \right.
\]
\[+ w_{12}^{P} \left(\frac{f_{12}(X) - (-5.2)}{767 - (-5.2)}\right)^{p} \]
\[+ w_{13}^{P} \left(\frac{f_{13}(X) - (-52.6)}{89 - (-52.6)}\right)^{p} \]
\[+ w_{14}^{P} \left(\frac{f_{14}(X) - (-13)}{115.6 - (-13)}\right)^{p} \right]^{1/p}
\]

- Thus, problem (12) is obtained.

- In order to get numerical solutions, assume that $w_{11}^{P} = w_{12}^{P} = w_{13}^{P} = w_{14}^{P} = 0.25$ and $p = 2$.

| Table (3): PIS payoff table of problem (12), when $p=2$. |
\begin{tabular}{|c|c|c|c|c|c|}
\hline
Min. $d_{P}^{FLDM}$ & $d_{2}^{FLDM}$ & $d_{2}^{FLDM}$ & $x_1$ & $x_2$ & $x_3$ & $x_4$ \\
\hline
$\begin{align*}
Min. d_{2}^{FLDM} && 0.2166 & 0.2999 & 0 & 13 & 31.6442 & 18.3558 \\
Max. d_{2}^{FLDM} && 0.3046 & 0.2232 & 2.6 & 0 & 5.4222 & 26.5777 \\
\end{align*}$
\end{tabular}

\[d_{2}^{FLDM} = (0.2166, 0.2999), d_{2}^{FLDM} = (0.3046, 0.2999).

- Now, it is easy to compute (15):

$\begin{align*}
Maximize \gamma^{FLDM} \\
subject to \quad x_3 + x_4 \leq 50, \\
x_1 + x_2 + x_3 + x_4 \leq 66, \\
5x_1 + x_2 \leq 13,
\end{align*}$

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\[
\left( \frac{0.2232 - d^NIS_{FLDM}}{0.2232 - 0.2999} \right) \geq y_{FLDM},
\]

\[ x_1, x_2, x_3, x_4 \geq 0, \quad y_{FLDM} \in [0,1]. \]

The maximum “satisfactory level” \(y_{FLDM} = 0.9929\) is achieved for the solution \(X^{FLDM}_1 = 0\), \(X^{FLDM}_2 = 13\), \(X^{FLDM}_3 = 1\), \(X^{FLDM}_4 = 1\) and \((f_{11}, f_{12}, f_{13}, f_{14}) = (78, 117, 39, -26)\). Let the FLDM decide \(X^{FLDM}_1 = 0\) and \(X^{FLDM}_2 = 13\) with positive tolerance \(t^R = 0.5\) and \(d^L = 0.5\) \([21, 52, 58]\).

- Obtain PIS and NIS payoff tables for the SLDM of the LS-BL-LMOP Problem (24).

<table>
<thead>
<tr>
<th>Table (4) : PIS payoff table for the SLDM of problem (24)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Maximize</strong></td>
</tr>
<tr>
<td>952.6</td>
</tr>
<tr>
<td><strong>Minimize</strong></td>
</tr>
</tbody>
</table>

**PIS**: \(f^*_{SLDM} = (952.6, -50)\)

<table>
<thead>
<tr>
<th>Table (5) : NIS payoff table for the SLDM of problem (24)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Minimize</strong></td>
</tr>
<tr>
<td>-65</td>
</tr>
<tr>
<td><strong>Maximize</strong></td>
</tr>
</tbody>
</table>

**NIS**: \(f^*_{SLDM} = (-65, 63)\)

- Next, compute equation (19) and obtain the following equations:
  \[
d^PIS^T^L = \left[ w^p_{21} \left( \frac{428 - f_{11}(x)}{428 - (-50)} \right)^p + w^p_{22} \left( \frac{767 - f_{12}(x)}{767 - (-52)} \right)^p + w^p_{23} \left( \frac{13(x) - (-52.6)}{13(x) - (-52.6)} \right)^p + w^p_{24} \left( \frac{115.6 - f_{14}(x) (115.6 - (-13))}{115.6 - (-13)} \right)^p + w^p_{25} \left( \frac{50.6 - f_{21}(x)}{952.6 - (-50)} \right)^p + w^p_{26} \left( \frac{63 - f_{22}(x)}{63 - (-50)} \right)^p \right]^1/p\]

- Thus, problem (20) is obtained.

- In order to get numerical solutions, assume that \(w^p_{21} = 0.2\), \(w^p_{22} = 0.2\), \(w^p_{23} = 0.2\), \(w^p_{24} = 0.2\), \(w^p_{25} = 0.1\), \(w^p_{26} = 0.1\) and \(p = 2\).

<table>
<thead>
<tr>
<th>Table (6) : PIS payoff table of problem (20), when (p = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d^PIS^T^L)</td>
</tr>
<tr>
<td>Min. (d^PIS^T^L)</td>
</tr>
<tr>
<td>Max. (d^NIS^T^L)</td>
</tr>
</tbody>
</table>

- \(d^T^L = (0.1950, 0.1948), \quad d^T^L = (0.2561, 0.2393)\)

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Now, it is easy to compute (23):

\[
\text{Maximize } \gamma^\text{BL}_i \\
\text{subject to}
\]

\[
x_1 + x_2 + x_3 + x_4 \leq 66, \quad 5x_1 + x_2 \leq 13, \quad x_3 + x_4 \leq 50,
\]

\[
\begin{align*}
\frac{d_2^\text{PIS}_i(X) - 0.1950}{0.2561 - 0.1950} & \geq \gamma^\text{BL}_i, \\
\frac{0.1948 - d_2^\text{NIS}_i(X)}{0.1948 - 0.2393} & \geq \gamma^\text{BL}_i, \\
\frac{(0 + 0.5) - x_1}{0.5} & \geq \gamma^\text{BL}_i, \\
\frac{(13 + 0.5) - x_2}{0.5} & \geq \gamma^\text{BL}_i, \\
\end{align*}
\]

The maximum “satisfactory level” \( \gamma^\text{BL}_i = 1 \) is achieved for the solution \( X^\text{BL}_1 = 0 \), \( X^\text{BL}_2 = 13 \), \( X^\text{BL}_3 = 1 \), \( X^\text{BL}_4 = 1 \).

**VII. SUMMARY AND CONCLUDING REMARKS**

In this paper, a TOPSIS approach has been extended to solve ST-TL-IL MODM with block angular structure. The ST-TL-IL MODM problems with block angular structure using TOPSIS approach provides an effective way to find the compromise (Satisfactory) integer solution of such problems. In order to obtain a compromise (satisfactory) integer solution to the ST-TL-IL MODM problems with block angular structure using the proposed TOPSIS approach, modified formulas for the distance function from the PIS and the distance function from the NIS are proposed and modeled to include all objective functions of both the first and the second levels. Thus, the bi-objective problem is obtained which can be solved by using membership functions of fuzzy set theory to represent the satisfaction level for both criteria and obtain TOPSIS, compromise solution by a second-order compromise. The max-min operator is then considered as a suitable one to resolve the conflict between the new criteria (the shortest distance from the PIS and the longest distance from the NIS). An interactive TOPSIS algorithm for solving these problems are also proposed. It is based on the decomposition algorithm of ILSMODM problems with block angular structure via TOPSIS approach, [5]. This algorithm has a few features, (i) it combines both ST-TL-IL MODM problems with block angular structure and TOPSIS approach to obtain TOPSIS’s compromise integer solution of the problem, (ii) it can be efficiently coded, (iii) it was found that the decomposition based method generally met with better results than the traditional simplex-based methods. Essentially, the efficiency of the decomposition-based method increased sharply with the scale of the problem. An illustrative numerical example is given to demonstrate the proposed TOPSIS approach and the decomposition algorithm.

**REFERENCES**


J. F. Bard, Coordination of a multidivisional organization through two levels of management, Omega, 11(1983)457–468.


